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Paracompactness on supra topological spaces

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Abstract. In this article, we present the concept of supra paracompact spaces and study its basic properties. We elucidate its relationship with supra compact spaces and prove that the property of being a supra paracompact space is weakly hereditary and topological properties. Also, we provide some examples to show some results concerning paracompactness on topology are invalid on supra topology. Finally, we investigate some results related to the product space and projection map.

Keywords: Supra paracompact space, weakly hereditary property, topological property, supra topology.

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1. Introduction and preliminaries

Generalizing the properties of the bounded and closed subsets of R^n is the motivation of introducing compactness into topology. In 1944, Dieudonné [14] introduced a wider class of compact spaces, namely paracompact spaces. Sorgenfrey [25] and Stone [26] investigated the behaviors of paracompact spaces under the product spaces. Michael [21] gave some characterizations for paracompactness under the condition of regular topological spaces and showed how metrizability implies paracompactness. Dowker [15] generalized paracompact spaces by introducing the class of countably paracompact spaces. New generalizations of paracompact spaces were given using α-open, pre-open, semi-open and β-open sets by [10, 11, 19] and [1], respectively. In 2019, [24] utilized a bijective function as another technique to present new types of paracompact spaces.

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Mashhour et al. [20] initiated the concept of supra topological spaces and investigated its main properties. They put a starting point for studying separation axioms and continuity on supra topology. Motivation for supra topology come from the need to obtain various examples that satisfy some properties on the finite sets, for instance, the only topology on a finite set that satisfies a T_1-space is the discrete topology, whereas there are many different types of supra topologies that satisfy a condition of a T_1-space.

Several scholars, after emergence of supra topology, investigated some topological notions on supra topological spaces such as compact spaces [4–7], separation axioms [4, 23] and generalized open sets [3, 16]. Some published papers showed that many topological results become true on supra topologies, but some of them fail such as a compact T_2-space is regular and the distributive property of the interior (resp. closure) operator for the intersection (resp. union) between two sets. Overall, it can be observed that all topological properties which associated with the intersection condition do not remain valid on supra topological spaces. In [8, 9], supra topological structure and supra separation axioms have been recently studied in soft setting.

In regard to applications field, supra topology provides a framework which is general enough to solve practical problems and to model phenomena. In [18], the authors have discussed some digital problems using supra topological frames. The collections of semi open and regular sets, which forms a supra topology structure, were exploited to solve or remove obstacles on the digital domain as investigated in [2, 17].

Generalized topology is another generalization of topology, see [12, 13] for recent studies on generalized topology. Interested readers of solutions of some nonlinear integral equations with applications can see [22].

Our goal in this article is to introduce the concept of paracompactness on supra topological spaces. We discuss main properties and show its relationships with compactness. Also, we study some results which connect between supra paracompact spaces and some notions such as hereditary and topological properties, product space and projection map. To elucidate main obtained results, we provide several examples.

In what follows, we recall some definitions and results which are required to make this work self-contained.

Definition 1.1 [20] A sub collection μ of 2^X is called a supra topology on $X \neq \emptyset$ provided that it contains X and is closed under arbitrary union. A pair (X, μ) is called a supra topological space. Every member of μ is called a supra open set and its complement is called a supra closed set.

Henceforth, (X, μ) and (Y, ν) denote supra topological spaces.

Definition 1.2 [20] A map $f : (X, \mu) \to (Y, \nu)$ is said to be an $S^\star$-continuous if the inverse image of every supra open subset of Y is a supra open subset of X.

Definition 1.3 [4] Let A be a subset of a supra topological space (X, μ). The family $\mu_A = \{A \bigcap G : G \in \mu\}$ is called a supra relative topology on A. A pair (A, μ_A) is called a supra subspace of (X, μ).

Lemma 1.4 [4] Consider (A, μ_A) is a subspace of (X, μ). Then, $(\overline{U})_A = A \bigcap \overline{U}$ for each $U \subseteq A$, where $(\overline{U})_A$ is the supra closure operator in (A, μ_A).

Definition 1.5 A map $f : (X, \mu) \to (Y, \nu)$ is said to be:

(1) $S^\star$-open if the image of any supra open subset of X is a supra open subset of Y.
(2) $S^\star$-homeomorphism if it is bijective, $S^\star$-continuous and $S^\star$-open.

Definition 1.6 [4] (X, μ) is called supra compact provided that every supra open cover of X has a finite subcover.

Definition 1.7 [14] Let $\mathcal{U} = \{U_i : i \in I\}$ and $\mathcal{V} = \{V_j : j \in J\}$ be two covers of X. The collection $\mathcal{U}$ is said to be a refinement of $\mathcal{V}$ provided that for each $U_i \in \mathcal{U}$, there are some $V_j \in \mathcal{V}$ such that $U_i \subseteq V_j$.

Definition 1.8 [14] The collection $\mathcal{U} = \{U_i : i \in I\}$ of subsets of a space X is said to be a locally finite provided that each point $x \in X$ has a neighbourhood W such that $W \bigcap U_i \neq \emptyset$ for only finitely many i.

Proposition 1.9 [14] If the collection $\{U_i : i \in I\}$ is locally finite, then the collection $\{\overline{U_i} : i \in I\}$ is locally finite.

Proposition 1.10 For any sets G, H, U, V, we have $(G \times H)\bigcap(U \times V) = (G \bigcap H) \times (U \bigcap V)$.

Definition 1.11 [7] Let $\{(X_i, \mu_i) : i = 1, 2, ..., n\}$ be a collection of supra topological spaces. Then $\mathcal{B} = \prod_{i=1}^{n} \mu_i = \{\prod_{i=1}^{n} G_i : G_i \in \mu_i\}$ defines a base for a supra topology μ on $X = \prod_{i=1}^{n} X_i$. We call (X, μ) a finite product supra spaces.

Definition 1.12 The two maps:

$$\textstyle\prod_x : X \times Y \to X \text{ such that } \prod_x((x, y)) = x \text{ for each } (x, y) \in X \times Y \quad \text{and}$$

$$\textstyle\prod_y : X \times Y \to Y \text{ such that } \prod_y((x, y)) = y \text{ for each } (x, y) \in X \times Y$$

are called the projection maps.

2. Supra paracompact spaces

In this section, we define the concept of supra paracompact spaces and study its main properties in terms of hereditary and topological properties. We provide several examples to illustrate the obtained results.

Definition 2.1 (X, μ) is said to be supra paracompact if every supra open cover has a supra open locally finite refinement.

Proposition 2.2 Every supra compact space (X, μ) is supra paracompact.

Proof. Let $\mathcal{U} = \{U_i : i \in I\}$ be a supra open cover of (X, μ). By hypothesis, there exists a finite subcover of $\mathcal{U}$ such that $X = \bigcup_{i=1}^{n} U_i$. Obviously, $\{U_i : i = 1, 2, ..., n\}$ is a supra open, locally finite refinement for $\mathcal{U}$. Hence, (X, μ) is supra paracompact. ■

Corollary 2.3 Every finite supra topological space (X, μ) is supra paracompact.

The converse of the above proposition and corollary need not be true as it is illustrated in the following example.

***Example* 2.4** Let ν be the collection of all pairwise disjoint binary subsets of the real numbers set $\mathcal{R}$ and their unions. Then $\mu = \emptyset \bigcup \nu$ is a supra topology on $\mathcal{R}$. Now, $\mathcal{U} = \{\{x, y\} \in \nu : x, y \in \mathcal{R}\}$ be a supra open cover of (X, μ). Since $\mathcal{U}$ has not a finite subcover of $\mathcal{R}$, then $(\mathcal{R}, \mu)$ is not supra compact. But $\mathcal{U}$ itself express a supra open, locally finite refinement for any supra open cover of $(\mathcal{R}, \mu)$. Hence, $(\mathcal{R}, \mu)$ is supra paracompact.

The following example shows that the existence of a supra topological space which is not supra paracompact.

***Example* 2.5** Let $\mu = \{\emptyset, G \subseteq \mathcal{R}$ such that $1 \in G$ or $2 \in G\}$ be a supra topology on the set of real numbers $\mathcal{R}$. Suppose that $\mathcal{U}$ is an infinite supra open cover of $\mathcal{R}$. Then any supra open refinement of $\mathcal{U}$ contains an infinite supra open sets containing 1 or 2. So that, we can not find a locally finite refinement for a supra open cover $\mathcal{U}$. Hence, $(\mathcal{R}, \mu)$ is not supra paracompact.

Proposition 2.6 (X, μ) is supra compact iff every supra open cover of (X, μ) has a finite supra open refinement that cover X.

Proof. Necessity: It follows from Proposition 2.2.
Sufficiency: Let $\mathcal{U} = \{U_i : i \in I\}$ be a supra open cover of (X, μ). By hypothesis, there exists a finite supra open refinement $\mathcal{V} = \{V_j : j = 1, 2, ..., n\}$ such that $X = \bigcup_{j=1}^{n} V_j$. Now, we construct ξ by choosing only one $U_i \in \mathcal{U}$ for each V_j such that $V_j \subseteq U_i$. Obviously, ξ is a finite supra open cover of (X, μ). Hence, (X, μ) is supra compact. ■

Proposition 2.7 The union of two supra paracompact sets is supra paracompact.

Proof. Suppose that $\mathcal{W} = \{W_i : i \in I\}$ is a supra open cover of a set $A \bigcup B$. Then $\mathcal{W}$ is a supra open cover of the sets A and B. By hypothesis, $\mathcal{W}$ has a supra open, locally finite refinements $\mathcal{U}$ and $\mathcal{V}$ of A and B, respectively. Obviously, the union of two families of locally finite refinements is a locally finite refinement. So that, $\mathcal{U} \bigcup \mathcal{V}$ is a supra open, locally finite refinement of $A \bigcup B$. Hence, the desired result is proved. ■

Definition 2.8 A property is said to be:

(1) a weakly hereditary property if the property passes from a supra topological space to every supra closed subspace.
(2) a supra topological property if the property is preserved by an $S^{\star}$-homeomorphism map.

Proposition 2.9 Every supra closed subset of a supra paracompact space (X, μ) is supra paracompact.

Proof. Let $\mathcal{U} = \{U_i : i \in I\}$ be a supra open cover of a supra closed set F. Then $\mathcal{U} \bigcup F^c$ be a supra open cover of (X, μ). It follows from supra paracompactness of (X, μ) that $\mathcal{U} \bigcup F^c$ has a supra open, locally finite refinement of (X, μ). Therefore $\mathcal{U}$ has a supra open, locally finite refinement of F. Hence, F is supra paracompact. ■

Corollary 2.10 The intersection of a supra paracompact set and a supra closed set in (X, μ) is supra paracompact.

Corollary 2.11 The property of being a supra paracompact space is a weakly hereditary property.

To show that the converse of the above proposition need not be true in general, we give the next two examples.

***Example* 2.12** Let $\mu = \{\emptyset, X, \{a, b, x\}, \{a, b, y\}, \{a, x, y\}, \{b, x, y\}, \{a, b\}, \{x, y\}, \{a, x\}, \{b, y\}, \{x, b\}\}$ be a supra topology on $X = \{a, b, x, y\}$. Now, $\{a, b, x\}$ is a supra paracompact set, but it is not supra closed.

Remark 1 *It is well known on general topology that a subspace of a paracompact space need not be paracompact. So that, a subspace of a supra paracompact space need not be supra paracompact.*

The following two results were proved on general topology.

Theorem 2.13

(1) Every paracompact subset of a T_2-space is closed.
(2) Every paracompact Hausdorff space is normal.

The above two topological results are invalid on supra topology. To show this fact, consider Example 2.12. It can be checked that (X, μ) is supra T_2. Obviously, $\{a, b, x\}$ is a supra paracompact set, but it is not supra closed. In addition, the two supra closed sets $\{b\}$ and $\{a, y\}$ are disjoint. Since there do not exist two disjoint supra open sets such that one of them contains $\{b\}$ and the other contains $\{a, y\}$, then (X, μ) is not supra normal.

Theorem 2.14 The property of being a supra paracompact space is a supra topological property.

Proof. Let $f : (X, \mu) \to (Y, \nu)$ be an $S^\star$-homeomorphism map and let X be supra paracompact. To prove that Y is supra paracompact, suppose that $\Lambda = \{G_i : i \in I\}$ is a supra open cover of Y. Then $\Omega = \{f^{-1}(G_i) : i \in I\}$ is a supra open cover of X. By hypothesis, there is a supra open, locally finite refinement $\xi = \{U_j : j \in J\}$ of Ω. This means that for each $j \in J$, we have some $i \in I$ such that $U_j \subseteq f^{-1}(G_i)$; and for each $x \in X$, there is a supra neighborhood W of x such that $W \bigcap U_j \neq \emptyset$ for only finitely many j. Now, $\{f(U_j) : j \in J\}$ is a supra open refinement for Y. Since for each $y \in Y$, $f(W)$ is a supra neighborhood of y such that $f(W) \bigcap f(U_j) \neq \emptyset$ for only finitely many j. Thus $\{f(U_j) : j \in J\}$ is a supra open locally finite refinement for Y. This ends the proof that Y is supra paracompact. ■

Theorem 2.15 If (X, μ) and (Y, ν) are two supra topological spaces and $(X \times Y, \mu \times \nu)$ is their product supra space, then the projection maps $\pi_x : (X \times Y, \mu \times \nu) \to (X, \mu)$ and $\pi_y : (X \times Y, \mu \times \nu) \to (Y, \nu)$ are surjective, $S^\star$-continuous and $S^\star$-open.

Proof. We only prove the theorem in the case of π_x and the other case follows similar line.
It is clear that for each $x \in X$, there exists $(x, y) \in X \times Y$ such that $\pi_x((x, y)) = x$. Then π_x is surjective. To prove the $S^\star$-continuity of π_x, let G be a supra open subset of X. Then $\pi_x^{-1}(G) = G \times Y$. Obviously, $G \times Y$ is a supra open subset of $X \times Y$. Hence, π_x is $S^\star$-continuous. To prove the $S^\star$-openness of π_x, let W be a supra open subset of $X \times Y$. Then W is written as the union of the members of base of $\mu \times \nu$. In other words, $W = \bigcup_{i \in I, j \in J}\{U_i \times V_j : U_i \in \mu \text{ and } V_j \in \nu\}$. Now, $\pi_x(\bigcup_{i \in I, j \in J}(U_i \times V_j)) = \bigcup_{i \in I, j \in J}(\pi_x(U_i \times V_j)) = \bigcup_{i \in I} U_i$. This ends the proof that π_x is $S^\star$-open. ■

Corollary 2.16 Let (X, μ) and (Y, ν) be two supra topological spaces and $(X \times Y, \mu \times \nu)$ be their product supra space. Then we have the following two results.

(1) For each fixed point $y \in Y$, the subspace $X \times \{y\}$ of $(X \times Y, \mu \times \nu)$ is $S^\star$-homeomorphic to (X, μ).
(2) For each fixed point $x \in X$, the subspace $\{x\} \times Y$ of $(X \times Y, \mu \times \nu)$ is $S^\star$-homeomorphic to (Y, ν).

Proof.

(1) Consider $(X \times \{y\}, T_{\mu \times \nu})$ is a subspace of $(X \times Y, \mu \times \nu)$. Let a map $g_x : (X \times \{y\}, T_{\mu \times \nu}) \to (X, \mu)$ defined by $g_x((x, y)) = x$. It is obvious that g_x is bijective. To prove the $S^\star$-continuity of g_x, let G be a supra open subset of X. Then $g_x^{-1}(G) = G \times \{y\}$. Obviously, $G \times \{y\}$ is a supra open subset of $(X \times \{y\}, T_{\mu \times \nu})$. Hence, g_x is $S^\star$-continuous. To prove the $S^\star$-openness of g_x, let W be a supra open subset of $(X \times \{y\}, T_{\mu \times \nu})$. Then $W = (X \times \{y\}) \bigcap H$

for some supra open subsets H of $(X \times Y, \mu \times \nu)$. Now, It can be written $H = \bigcup_{i\in I, j\in J}\{U_i \times V_j : U_i \in \mu \text{ and } V_j \in \nu\}$. Consequently, $W = (X \times \{y\})\bigcap[\bigcup_{i\in I,j\in J}\{U_i \times V_j : U_i \in \mu \text{ and } V_j \in \nu\}] = \bigcup_{i\in I,j\in J}[(X \times \{y\})\bigcap(U_i \times V_j)] = \bigcup_{i\in I,j\in J}[(X\bigcap U_i) \times (\{y\}\bigcap V_j)] = \bigcup_{i\in I,j\in J}[U_i \times (\{y\}\bigcap V_j)]$. Now, $g_x(W) = g_x(\bigcup[U_i \times (\{y\}\bigcap V_j)]) = \bigcup_{i\in I,j\in J}(g_x([U_i \times (\{y\}\bigcap V_j)])$. This means:

$$g_x(W) = \begin{cases} \emptyset & : \ y \notin V_j \\ \bigcup\limits_{for\ some\ U_i} U_i & : \ y \in V_j \end{cases}$$

This means that $g_x(W)$ is a supra open subset of $(X \times Y, \mu \times \nu)$. Thus, $g_x(W)$ is $S^{\star}$-open. Hence, the desired result is proved.

(2) Following arguments similar to those given in (1), the result holds.

■

Corollary 2.17 If (X, μ) is supra paracompact, then $X \times \{y\}$ is supra paracompact.

Remark 2 *It is well known on general topology that the lower limit topological space $(\mathcal{R}, \tau_l)$ is paracompact. However, the product of two lower limit topological spaces is not paracompact. So that, the product of supra paracompact spaces need not be supra paracompact.*

3. Conclusion

In the last few years, the interest to the study of supra topologies has increased and a number of articles on this topic has been published. The concepts presented in supra topological spaces have been defined in analogy with topological spaces. By studying supra topology, we can improve some properties of a topology which defined on a finite set as explained in the second paragraph of introduction section and provide a convenient model to describe many real life problems as investigated in [2, 18]. In this work, we have introduced the concept of paracompactness on supra topological spaces. We have presented its main properties with the help of some interesting examples, in particular, hereditary and topological properties. In an upcoming research, we are going to discuss an application of supra topology on the information system.

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