Somehow-connectedness and somewhat-continuity in the product space

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Abstract. In this paper, the concept of somewhat-connected space will be introduced and characterized. Its connection with the other well-known concepts such as the classical connectedness, the $\omega$-connectedness, and the $\omega$-connectedness will be determined. Moreover, the concept of somewhat-continuous function from an arbitrary topological space into the product space will be characterized.

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1. Introduction and preliminaries

In literature, substituting several concepts in topology with concepts acquiring either of weaker or stronger properties is often studied. The first attempt was done by Levine [24] when he introduced the concepts of semi-open set, semi-closed set, and semi-continuity of a function. Several mathematicians then became interested in introducing other topological concepts which can replace the concept of open set, closed set, and continuity of a function.

In 1968, Velicko [27] introduced the concepts of $\theta$-continuity between topological spaces and subsequently defined the concepts of $\theta$-closure and $\theta$-interior of a subset of topological space. The concept of $\theta$-open sets and its related topological concepts had been deeply studied and investigated by numerous authors, see [1, 7, 8, 15, 16, 20–22, 25, 26].

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Let $(X, T)$ be a topological space and $A \subseteq X$. The $\theta$-closure and $\theta$-interior of $A$ are, respectively, denoted and defined by $Cl_{\theta}(A) = \{ x \in X : Cl(U) \cap A \neq \emptyset \text{ for every open set } U \text{ containing } x \}$ and $Int_{\theta}(A) = \{ x \in X : Cl(U) \subseteq A \text{ for some open set } U \text{ containing } x \}$, where $Cl(U)$ is the closure of $U$ in $X$. A subset $A$ of $X$ is $\theta$-closed if $Cl_{\theta}(A) = A$ and $\theta$-open if $Int_{\theta}(A) = A$. Equivalently, $A$ is $\theta$-open if and only if $X \setminus A$ is $\theta$-closed.

In 1971, Hoyle and Gentry [19] introduced the class of somewhat-continuous functions and somewhat-open functions. The somewhat-continuous functions, which are generalization of continuity requiring nonempty inverse images of open sets to have nonempty interiors instead of being open, have proved to be very useful in topology. Since then, the concepts of somewhat-interior and somewhat-closure of a subset of a topological space have been subsequently defined and the concept of somewhat-open and somewhat-closed sets have been used to characterize somewhat-continuity, see [5, 6].

A subset $U$ of a space $X$ is said to be somewhat-open if $U = \emptyset$ or if there exists $x \in U$ and an open set $V$ such that $x \in V \subseteq U$. A set is called somewhat-closed if its complement is somewhat open. Denote by $T_{sw}$, the collection of all somewhat-open sets in $X$. Let $A$ be a subset of a space $X$. The somewhat-closure and somewhat-interior of $A$ are, respectively, denoted and defined by $swCl(A) = \cap \{ F : F \text{ is somewhat-closed and } A \subseteq F \}$ and $swInt(A) = \cup \{ U \subseteq A : U \text{ is somewhat-open} \}$.

Let $X$ and $Y$ be topological spaces. A function $f : X \to Y$ is said to be

(i) somewhat-open provided that for every open set $U$ of $X$ such that $U \neq \emptyset$, there exists an open set $V$ of $Y$ such that $V \neq \emptyset$ and $V \subseteq f(U)$;

(ii) somewhat-closed if for every closed set $F$ of $X$ such that $F \neq \emptyset$, there exists a closed set $G$ of $Y$ such that $\emptyset \neq G \subseteq f(F)$; and

(iii) somewhat-continuous if for every open set $V$ of $Y$ such that $f^{-1}(V) \neq \emptyset$ there exists an open set $U$ of $X$ such that $U \neq \emptyset$ and $U \subseteq f^{-1}(V)$.

In 1982, Hdeib [18] introduced the concepts of $\omega$-open and $\omega$-closed sets and $\omega$-closed mappings on a topological space. He showed that $\omega$-closed mappings are strictly weaker than closed mappings and also showed that the Lindelöf property is preserved by counter images of $\omega$-closed mappings with Lindelöf counter image of points. The concepts of $\omega$-open sets and its corresponding topological concepts had been studied in several papers, see [3, 4, 9–14, 23].

In 2010, Ekici et al. [17] introduced the concepts of $\omega_{\theta}$-open and $\omega_{\theta}$-closed sets on a topological space. They showed that the family of all $\omega_{\theta}$-open sets in a topological space $X$ forms a topology on $X$. They also introduced the notions of $\omega_{\theta}$-interior and $\omega_{\theta}$-closure of a subset of a topological space.

A point $x$ of a topological space $X$ is called a condensation point of $A \subseteq X$ if for each open set $G$ containing $x$, $G \cap A$ is uncountable. A subset $B$ of $X$ is $\omega$-closed if it contains all of its condensation points. The complement of $B$ is $\omega$-open. Equivalently, a subset $U$ of $X$ is $\omega$-open (resp., $\omega_{\theta}$-open) if and only if for each $x \in U$, there exists an open set $O$ containing $x$ such that $O \setminus U$ (resp., $O \setminus Int_{\theta}(U)$) is countable. A subset $B$ of $X$ is $\omega_{\theta}$-closed if its complement $X \setminus B$ is $\omega$-open. The $\omega$-closure (resp., $\omega_{\theta}$-closure) and $\omega$-interior (resp., $\omega_{\theta}$-interior) of $A \subseteq X$ are, respectively, denoted and defined by $Cl_{\omega}(A) = \cap \{ F : F \text{ is an } \omega \text{-closed set containing } A \}$ (resp., $Cl_{\omega_{\theta}}(A) = \cap \{ F : F \text{ is an } \omega_{\theta} \text{-closed set containing } A \}$) and $Int_{\omega}(A) = \cup \{ G : G \text{ is an } \omega \text{-open set contained in } A \}$ (resp., $Int_{\omega_{\theta}}(A) = \cup \{ G : G \text{ is an } \omega_{\theta} \text{-open set contained in } A \}$).

It is worth noting that $A \subseteq Cl_{\omega_{\theta}}(A)$ (resp., $A \subseteq Cl_{\theta}(A)$, $A \subseteq Cl_{\omega}(A)$ and $Int_{\omega_{\theta}}(A) \subseteq A$ (resp., $Int_{\theta}(A) \subseteq A$, $Int_{\omega}(A) \subseteq A$). Let $T_{\omega_{\theta}}$ (resp., $T_{\theta}$, $T_{\omega}$) be the family of all $\omega_{\theta}$-open (resp., $\theta$-open, $\omega$-open) subsets of a topological space $X$. Since $T_{\omega_{\theta}}$ (resp., $T_{\theta}$, $T_{\omega}$)
is a topology on $X$ for any set $A \subseteq X$, $\text{Int}_\omega(A)$ (resp., $\text{Int}_\theta(A)$, $\text{Int}_\omega(A)$) is $\omega_\theta$-open (resp., $\theta$-open, $\omega$-open) and the largest $\omega_\theta$-open (resp., $\theta$-open, $\omega$-open) set contained in $A$. Moreover, for any set $A \subseteq X$, $\text{Cl}_\omega(A)$ (resp., $\text{Cl}_\theta(A)$, $\text{Cl}_\omega(A)$) is $\omega_\theta$-closed (resp., $\theta$-closed, $\omega$-closed) and the smallest $\omega_\theta$-closed (resp., $\theta$-closed, $\omega$-closed) set containing $A$.

A topological space $X$ is said to be somewhat-connected (resp., $\theta$-connected, $\omega$-connected, $\omega_\theta$-connected) if $X$ cannot be written as the union of two nonempty disjoint somewhat-open (resp., $\theta$-open, $\omega$-open, $\omega_\theta$-open) sets. Otherwise, $X$ is somewhat-disconnected (resp., $\theta$-disconnected, $\omega$-disconnected, $\omega_\theta$-disconnected). A subset $B$ of $X$ is somewhat-connected (resp., $\theta$-connected, $\omega$-connected, $\omega_\theta$-connected) if it is somewhat-connected (resp., $\theta$-connected, $\omega$-connected, $\omega_\theta$-connected) as a subspace of $X$.

A function $f$ from a topological space $X$ to another topological space $Y$ is said to be

(i) $\omega_\theta$-open (resp., $\theta$-open, $\omega$-open) if $f(G)$ is $\omega_\theta$-open (resp., $\theta$-open, $\omega$-open) in $Y$ for every open set $G$ in $X$; and

(ii) $\omega_\theta$-$\text{closed}$ (resp., $\theta$-$\text{closed}$, $\omega$-$\text{closed}$) if $f(G)$ is $\omega_\theta$-$\text{closed}$ (resp., $\theta$-$\text{closed}$, $\omega$-$\text{closed}$) in $Y$ for every closed set $G$ in $X$.

Let $A$ be an indexing set and $\{Y_\alpha : \alpha \in A\}$ be a family of topological spaces. For each $\alpha \in A$, let $T_\alpha$ be the topology on $Y_\alpha$. The Tychonoff topology on $\Pi\{Y_\alpha : \alpha \in A\}$ is the topology generated by a subbase consisting of all sets $p_\alpha^{-1}(U_\alpha)$, where the projection map $p_\alpha : \Pi\{Y_\alpha : \alpha \in A\} \to Y_\alpha$ is defined by $p_\alpha(\langle y_\beta \rangle) = y_\alpha$. $U_\alpha$ ranges over all members of $T_\alpha$, and $\alpha$ ranges over all elements of $A$. Corresponding to $U_\alpha \subseteq Y_\alpha$, denote $p_\alpha^{-1}(U_\alpha)$ by $\langle U_\alpha \rangle$. Similarly, for finitely many indices $\alpha_1, \alpha_2, \ldots, \alpha_n$, and sets $U_{\alpha_1} \subseteq Y_{\alpha_1}, U_{\alpha_2} \subseteq Y_{\alpha_2}, \ldots, U_{\alpha_n} \subseteq Y_{\alpha_n}$, the subset

$$\langle U_{\alpha_1} \rangle \cap \langle U_{\alpha_2} \rangle \cap \cdots \cap \langle U_{\alpha_n} \rangle = p_{\alpha_1}^{-1}(U_{\alpha_1}) \cap p_{\alpha_2}^{-1}(U_{\alpha_2}) \cap \cdots \cap p_{\alpha_n}^{-1}(U_{\alpha_n})$$

is denoted by $\langle U_{\alpha_1}, U_{\alpha_2}, \ldots, U_{\alpha_n} \rangle$. We note that for each open set $U_\alpha$ subset of $Y_\alpha$, $\langle U_\alpha \rangle = p_\alpha^{-1}(U_\alpha) = U_\alpha \times \Pi_{\beta \neq \alpha} Y_\beta$. Hence, a basis for the Tychonoff topology consists of sets of the form $\langle B_{\alpha_1}, B_{\alpha_2}, ..., B_{\alpha_n} \rangle$, where $B_{\alpha_i}$ is open in $Y_{\alpha_i}$ for every $i \in K = \{1, 2, ..., k\}$.

Now, the projection map $p_\alpha : \Pi\{Y_\alpha : \alpha \in A\} \to Y_\alpha$ is defined by $p_\alpha(\langle y_\beta \rangle) = y_\alpha$ for each $\alpha \in A$. It is known that every projection map is a continuous open surjection. Also, it is well known that a function $f$ from an arbitrary space $X$ into the Cartesian product $Y$ of the family of spaces $\{Y_\alpha : \alpha \in A\}$ with the Tychonoff topology is continuous if and only if each coordinate function $p_\alpha \circ f$ is continuous, where $p_\alpha$ is the $\alpha$-th coordinate projection map.

In this paper, we revisit the concept of somewhat-open sets and characterize its related topological concepts such as somewhat-connected space and somewhat-continuous functions from an arbitrary topological space into the product space.

2. Somewhat-open and somewhat-closed functions

In this section, we investigate the connection of somewhat-open (resp., somewhat-closed) function to the other well-known functions such as the classical open, $\theta$-open, $\omega$-open, and $\omega_\theta$-open (resp., closed, $\theta$-closed, $\omega$-closed, $\omega_\theta$-closed) functions. We also give some characterizations of a somewhat-open function. Throughout, if no confusion arises, let $X$ and $Y$ be topological spaces.

Remark 1 The arbitrary union of somewhat-open sets is somewhat-open, but the finite intersection of somewhat-open sets is not necessarily somewhat-open. This means that
Consider topological space \((X, T)\) where \(X = \{1, 2, 3\}\) and \(T = \{\emptyset, X, \{1\}, \{2\}, \{1, 2\}\}\). Then \(\{1, 3\}\) and \(\{2, 3\}\) are both somewhat-open sets but their intersection, which is \(\{3\}\), is not somewhat-open.

**Remark 2** Let \(X\) be a topological space and \(A, B \subseteq X\).

(i) \(sw\text{Int}(A)\) is somewhat-open and \(sw\text{Int}(A) \subseteq A\).

(ii) \(sw\text{Cl}(A)\) is somewhat-closed and \(A \subseteq sw\text{Cl}(A)\).

(iii) \(sw\text{Int}(A)\) is the largest somewhat-open set contained in \(A\);

(iv) If \(A \subseteq B\), then \(sw\text{Int}(A) \subseteq sw\text{Int}(B)\);

(v) \(x \in sw\text{Int}(A)\) if and only if there exists a somewhat-open set \(U\) containing \(x\) such that \(U \subseteq A\);

(vi) \(A\) is somewhat-open if and only if \(A = sw\text{Int}(A)\);

(vii) \(sw\text{Int}(sw\text{Int}(A)) = sw\text{Int}(A)\);

(viii) \(sw\text{Int}(A \cap B) \subseteq sw\text{Int}(A) \cap sw\text{Int}(B)\);

(ix) \(sw\text{Cl}(A)\) is the smallest somewhat-closed set containing \(A\);

(x) \(A \subseteq B\) implies that \(sw\text{Cl}(A) \subseteq sw\text{Cl}(B)\);

(xi) \(sw\text{Cl}(sw\text{Cl}(A)) = sw\text{Cl}(A)\);

(xii) \(sw\text{Cl}(A) \cup sw\text{Cl}(B) \subseteq sw\text{Cl}(A \cup B)\);

The following three results can proved by modifying the construction given in [1], [2], and [17], respectively.

**Theorem 2.1** Let \(f : X \rightarrow Y\) be a function. If \(f\) is \(\theta\)-open (resp., \(\theta\)-closed), then \(f\) is open (resp., closed), but not conversely.

**Theorem 2.2** Let \(f : X \rightarrow Y\) be a function. If \(f\) is open (resp., closed), then \(f\) is \(\omega\)-open (resp., \(\omega\)-closed), but not conversely.

**Theorem 2.3** Let \(f : X \rightarrow Y\) be a function.

(i) If \(f\) is \(\theta\)-open (resp., \(\theta\)-closed), then \(f\) is \(\omega\theta\)-open (resp., \(\omega\theta\)-closed), but not conversely.

(ii) If \(f\) is \(\omega\theta\)-open (resp., \(\omega\theta\)-closed), then \(f\) is \(\omega\)-open (resp., \(\omega\)-closed), but not conversely.

**Theorem 2.4** Let \(X\) be topological space and \(A \subseteq X\). If \(A\) is open (resp., closed) then \(A\) is somewhat-open (resp., somewhat-closed), but not conversely.

**Proof.** Suppose that \(A\) is open. It is immediate when \(A = \emptyset\). Now, assume that \(A \neq \emptyset\) and let \(x \in A\). Then \(x \in A \subseteq A\), implying that \(A\) is somewhat-open. If \(A\) is closed then \(X \setminus A\) is somewhat-open. By the definition of a somewhat-closed set, \(X \setminus (X \setminus A) = A\) is somewhat-closed.

To show that the converses do necessarily hold, consider the topological space \((X, T)\), where \(X = \{a, b, c\}\) and \(T = \{\emptyset, X, \{a\}, \{b\}, \{a,b\}\}\). Let \(U = \{a, c\}\). Then \(U\) is not an open set. Note that \(a \in U\) and \(\{a\} \in T\). Hence, \(U\) is somewhat-open. Moreover, \(\{b\}\) is somewhat-closed but not closed.

**Theorem 2.5** Let \(f : X \rightarrow Y\) be a function. If \(f\) is open (resp., closed), then \(f\) is somewhat-open (resp., somewhat-closed), but not conversely.

**Remark 3** The following diagrams hold for a function \(f : X \rightarrow Y\).

**Remark 4** Somewhat-open (resp., closed) set and \(\omega\)-open (resp., closed) set are two independent notions.
To see this, first we will construct an $\omega$-open (resp., $\omega$-closed) set which is not somewhat-open (resp., somewhat-closed). Second, we will construct a somewhat-open (resp., somewhat-closed) set which is not an $\omega$-open (resp., $\omega$-closed) set. Now, consider the topological space $X = \{a, b, c\}$ with the topology $T = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$. We will show that $A = \{c\}$ is $\omega$-open but not somewhat-open in $(X, T)$. Since $X$ is countable, $A$ is $\omega$-open, but not somewhat-open since we cannot find open subset $V$ containing $c$ such that $V \subseteq A$. Moreover, $X \setminus A = \{a, b\}$ is $\omega$-closed but not somewhat-closed. For the second part, consider $\mathbb{R}$ as the real line with the topology $T = \{\mathbb{R}, \emptyset, \mathbb{N} \cup \{0\}\}$. We will show that $Z$ is somewhat-open but not $\omega$-open set in $(\mathbb{R}, T)$. Note that, $0 \in Z, \mathbb{N} \cup \{0\}$ is an open set containing 0 and $\mathbb{N} \cup \{0\} \subseteq Z$. Hence, $Z$ is somewhat-open. Next, we will show that $Z$ is not $\omega$-open. Proceeding by contradiction, suppose that $Z$ is $\omega$-open. Since $-1 \in Z$ then $\mathbb{N} \cup \{0\}$ does not contain $-1$. Thus, $\mathbb{R}$ is the only open set containing $-1$ which implies that $\mathbb{R} \setminus Z$ is countable which is a contradiction. Therefore, $Z$ is not $\omega$-open. Furthermore, $\mathbb{R} \setminus Z$ is somewhat-closed but not $\omega$-closed.

**Remark 5** Somewhat-open (resp., closed) set and $\omega_0$-open (resp., closed) set are two independent notions.

To see this, consider the topological space $X = \{x, y, z\}$ with the topology $T = \{\emptyset, X, \{x\}, \{z\}, \{x, z\}\}$. We will show that $A = \{y\}$ is $\omega_0$-open but not somewhat-open in $(X, T)$. Since $X$ is countable, $A$ is $\omega_0$-open. But $A$ is not somewhat-open since we cannot find open subset $V$ containing $y$ such that $V \subseteq A$. Moreover, $\{x, z\}$ is $\omega_0$-closed but not somewhat closed.

On the other hand, suppose that somewhat-open (resp., somewhat-closed) implies $\omega_0$-open (resp., $\omega_0$-closed). Then somewhat-open (resp., somewhat-closed) implies $\omega$-open (resp., $\omega$-closed), which is a contradiction.

The next result presents a characterization of somewhat-open functions.

**Theorem 2.6** Let $X$ and $Y$ be topological spaces and $f : X \rightarrow Y$ be a function. Then the following statements are equivalent:

(i) $f$ is somewhat-open on $X$.

(ii) $f(O)$ is somewhat-open in $Y$ for every open set $O$ in $X$.

(iii) $f(\text{Int}(A)) \subseteq \text{swInt}(f(A))$ for every $A \subseteq X$.

(iv) $f$ sends each member of a basis for $X$ to a somewhat-open set in $Y$.

(v) For each $p \in X$ and every open set $O$ in $X$ containing $p$, there exists a somewhat-open set $W$ in $Y$ such that $f(p) \in \text{swInt}(W) \subseteq f(O)$.

**Proof.** (i)$\Rightarrow$(ii): Suppose that $f$ is somewhat-open and let $O$ be an open set in $X$. If $O = \emptyset$ then $f(O) = \emptyset$ and so $f(O)$ is somewhat-open in $Y$. Now, assume that $O \neq \emptyset$. Then, by the definition of a somewhat-open function, there exists an open set $V$ in $Y$ such that $V \neq \emptyset$ and $V \subseteq f(O)$. Thus $f(O)$ is somewhat-open in $Y$.
Let $\text{Theorem 2.7}$ that $f$ is somewhat-open in $Y$ for every open set $O$ in $X$ and let $A \subseteq X$. Then $f(\text{Int}(A)) \subseteq f(A)$. Since $\text{Int}(A)$ is open in $X$, $f(\text{Int}(A))$ is somewhat-open in $Y$. Thus, $f(\text{Int}(A)) \subseteq \text{swInt}(f(A))$ since $\text{swInt}(f(A))$ is the largest somewhat-open set contained in $f(A)$.

(iii) $\Rightarrow$ (iv): Let $B$ be a basic open set in $X$. Then $f(B) = f(\text{Int}(B))$. By assumption, $f(B) = f(\text{Int}(B)) \subseteq \text{swInt}(f(B)) \subseteq f(B)$. Hence, $f(B) = \text{swInt}(f(B))$. By Remark 2 (vi), $f(B)$ is somewhat-open in $Y$.

(iv) $\Rightarrow$ (v): Let $O$ be open in $X$ and let $p \in O \subseteq X$. Since $O$ is open, there exists a basic open set $B$ containing $p$ such that $B \subseteq O$. This implies that $f(p) \in f(B) \subseteq f(O)$. By assumption, there exists a somewhat-open set $W$ in $Y$ such that $f(p) \in W \subseteq f(B)$. By Remark 2 (v), $f(p) \in \text{swInt}(W) \subseteq f(O)$.

(v) $\Rightarrow$ (i): Let $O$ be open in $X$ and $y \in f(O)$. Then there exists $x \in O$ such that $f(x) = y$. By assumption, there exists a somewhat-open set $W$ in $Y$ containing $y$ such that $W = \text{swInt}(W) \subseteq f(O)$. Thus, $f$ is somewhat-open on $X$.

Theorem 2.7 Let $f : X \to Y$ be a bijective function. Then $f$ is somewhat-open on $X$ if and only if $f(G)$ is somewhat-closed for every closed set $G$ in $X$.

3. Somewhat-continuity in the product space

In this section, a characterization of a somewhat-continuous function from an arbitrary topological space into the product space will be presented.

We shall give first a characterization of a somewhat-continuous function from a topological space to another topological space.

Theorem 3.1 Let $X$ and $Y$ be topological spaces and $f : X \to Y$ be a function. Then the following statements are equivalent:

(i) $f$ is somewhat-continuous on $X$.
(ii) $f^{-1}(G)$ is somewhat-open in $X$ for each open subset $G$ of $Y$.
(iii) $f^{-1}(F)$ is somewhat-closed in $X$ for each closed subset $F$ of $Y$.
(iv) $f^{-1}(B)$ is somewhat-open in $X$ for each (subbasic) basic open set $B$ in $Y$.
(v) For every $p \in X$ and every open set $V$ of $Y$ containing $f(p)$, there exists a somewhat-open set $U$ containing $p$ such that $f(U) \subseteq V$.
(vi) $f(\text{swCl}(A)) \subseteq \text{Cl}(f(A))$ for each $A \subseteq X$.
(vii) $\text{swCl}(f^{-1}(B)) \subseteq f^{-1}(\text{Cl}(B))$ for each $B \subseteq Y$.

Proof. Statements (i), (ii), (iii), and (v) are equivalent by [6, Theorem 3.2].

(ii) $\Rightarrow$ (iv): Suppose that $f^{-1}(B)$ is somewhat-open in $X$ for each $B \in B$ where $B$ is a basis for the topology in $Y$. Let $G$ be an open set in $Y$. Then $G = \cup\{B : B \in B^*\}$, for some $B^* \subseteq B$. It follows that $f^{-1}(G) = \cup\{f^{-1}(B) : B \in B^*\}$. Since the arbitrary union of somewhat-open sets is somewhat-open, $f^{-1}(G)$ is somewhat-open in $X$.

(v) $\Rightarrow$ (vii): Let $A \subseteq X$ and $p \in \text{swCl}(A)$. Let $G$ be an open set of $Y$ containing $f(p)$. Then, by assumption, there exists a somewhat-open set $O$ of $X$ containing $p$ such that $f(O) \subseteq G$. Since $p \in \text{swCl}(A)$, $O \cap A \neq \emptyset$, by [6, Theorem 3.8 (e)]. Thus, $\emptyset \neq f(O \cap A) \subseteq f(O) \cap f(A) \subseteq G \cap f(A)$. This implies that $f(p) \in \text{Cl}(f(A))$. Hence, $f(\text{swCl}(A)) \subseteq \text{Cl}(f(A))$.

(vii) $\Rightarrow$ (v): Let $B \subseteq Y$ and let $A = f^{-1}(B) \subseteq X$. By assumption, $f(\text{swCl}(A)) \subseteq \text{Cl}(f(A))$. Hence, $\text{swCl}(f^{-1}(B)) \subseteq f^{-1}(f(\text{swCl}(A))) \subseteq f^{-1}(\text{Cl}(f(A))) = f^{-1}(\text{Cl}(B))$. Thus, $\text{swCl}(f^{-1}(B)) \subseteq f^{-1}(\text{Cl}(B))$. 

\[\blacksquare\]
(vii) $\Rightarrow$ (iii): Let $F$ be a closed subset of $Y$. Then, $Cl(F) = F$. By assumption,

$$swCl(f^{-1}(F)) \subseteq f^{-1}(Cl(F)) = f^{-1}(F).$$

Since $f^{-1}(F) \subseteq swCl(f^{-1}(F))$, it follows that $swCl(f^{-1}(F)) = f^{-1}(F)$ and so $f^{-1}(F)$ is somewhat-closed in $X$. 

Theorem 3.2 Let $O$ be a nonempty somewhat-open set in the product space $Y = \prod\{Y_\alpha : \alpha \in A\}$. Then $p_\alpha(O) = Y_\alpha$ for all but at most finitely many $\alpha$ and $p_\alpha(O)$ is somewhat-open in $Y_\alpha$ for every $\alpha \in A$.

Proof. Let $x \in O$. Then there exists a basic open set $U = \langle U_{\alpha_1}, \ldots, U_{\alpha_n} \rangle$ such that $x \in U \subseteq O$. It follows that $p_\alpha(U) \subseteq p_\alpha(O)$ for every $\alpha \in A$. Note that $p_\alpha(U) = p_\alpha(\langle U_{\alpha_1}, \ldots, U_{\alpha_n} \rangle) = Y_\alpha$ for each $\alpha \notin \{\alpha_1, \ldots, \alpha_n\}$. Hence $p_\alpha(O) = Y_\alpha$ for all but at most a finite number of indices in $A$.

Next, let $\alpha \in A$. Then $p_\alpha(O) = Y_\alpha$ or $p_\alpha(O) \neq Y_\alpha$. If $p_\alpha(O) = Y_\alpha$, then $p_\alpha(O)$ is somewhat-open in $Y_\alpha$. Suppose that $p_\alpha(O) \neq Y_\alpha$. Since $O \neq \emptyset$, $p_\alpha(O) \neq \emptyset$. Since $O$ is a nonempty somewhat-open, there exists $y \in O$ and open set $V = \langle V_{\alpha_1}, \ldots, V_{\alpha_n} \rangle$ such that $y \in V = \langle V_{\alpha_1}, \ldots, V_{\alpha_n} \rangle \subseteq O$. Hence, there exists $p_\alpha(V) \subseteq p_\alpha(O)$ and an open set $p_\alpha(V) = V_\alpha$ such that $p_\alpha(y) \in V_\alpha \subseteq p_\alpha(O)$. This shows that $p_\alpha(O)$ is somewhat-open in $Y_\alpha$.

Corollary 3.3 Let $X$ be a topological space, $Y = \prod\{Y_\alpha : \alpha \in A\}$ a product space, and $f_\alpha : X \to Y_\alpha$ a function for each $\alpha \in A$. Let $f : X \to Y$ be the function defined by $f(x) = \langle f_\alpha(x) \rangle$. Then, $f$ is somewhat-continuous on $X$ if and only if each $f_\alpha$ is somewhat-continuous for each $\alpha \in A$.

Theorem 3.4 Let $Y = \prod\{Y_\alpha : \alpha \in A\}$ be a product space. Let $S = \{\alpha_1, \ldots, \alpha_n\}$ be a finite subset of $A$ and $\emptyset \neq O_\alpha_1 \subseteq Y_\alpha$ for each $\alpha \in A$. Then $O = \langle O_{\alpha_1}, \ldots, O_{\alpha_n} \rangle$ is somewhat-open in $Y$ if and only if each $O_\alpha_i$ is somewhat-open in $Y_\alpha$.

Proof. Suppose that $O = \langle O_{\alpha_1}, \ldots, O_{\alpha_n} \rangle$ is somewhat-open in $Y$ and $\emptyset \neq O_\alpha \subseteq Y_\alpha$ for each $\alpha \in A$. Choose $a_\alpha \in O_\alpha$. Suppose that $f_\alpha(x) = a_\alpha$. Since $O$ is somewhat-open in $Y$, there exists an open set $U = \langle U_{\alpha_1}, \ldots, U_{\alpha_n} \rangle$ such that $x = \langle a_\alpha \rangle \in \langle U_{\alpha_1}, \ldots, U_{\alpha_n} \rangle \subseteq \langle O_{\alpha_1}, \ldots, O_{\alpha_n} \rangle$. Thus, $p_\alpha(\langle a_\alpha \rangle) \subseteq p_\alpha(\langle U_{\alpha_1}, \ldots, U_{\alpha_n} \rangle)$ implying that $a_\alpha \in U_\alpha \subseteq O_\alpha$. Hence, each $O_\alpha_i$ is somewhat-open in $Y_\alpha$.

Conversely, suppose that each $O_\alpha_i$ is somewhat-open in $Y_\alpha$. Then, $O \neq \emptyset$. Hence, there exists $x = \langle a_\alpha \rangle \in O$. Then $a_{\alpha_i} \in O_{\alpha_i}$ for each $\alpha_i \in S$. Hence, there exists an open set $V_{\alpha_i}$ such that $a_{\alpha_i} \in V_{\alpha_i} \subseteq O_{\alpha_i}$. Then $V$ is open in $Y$ such that $x \in V \subseteq O$. Thus, $O$ is somewhat-open in $Y$.

Theorem 3.5 Let $X = \prod\{X_\alpha : \alpha \in A\}$ and $Y = \prod\{Y_\alpha : \alpha \in A\}$ be product spaces and $f_\alpha : X_\alpha \to Y_\alpha$ be a function for each $\alpha \in A$. If each $f_\alpha$ is somewhat-continuous on $X_\alpha$ then the function $f : X \to Y$ defined by $f((x_\alpha)) = \langle f_\alpha(x_\alpha) \rangle$ is somewhat-continuous on $X$.

Proof. Let $\langle V_\alpha \rangle$ be a subbasic open set in $Y$. Then $f^{-1}(\langle V_\alpha \rangle) = \langle f_\alpha^{-1}(V_\alpha) \rangle$. Since $f_\alpha$ is somewhat-continuous, $f_\alpha^{-1}(V_\alpha)$ is somewhat-open in $X_\alpha$. If $f_\alpha^{-1}(V_\alpha) = \emptyset$ for some $\alpha \in A$, then $f^{-1}(\langle V_\alpha \rangle) = \emptyset$, so that $f^{-1}(\langle V_\alpha \rangle)$ is somewhat-open in $X$. Assume that $f_\alpha^{-1}(V_\alpha) \neq \emptyset$ for every $\alpha \in A$. Then $f^{-1}(\langle V_\alpha \rangle) \neq \emptyset$. Let $x = \langle X_\alpha \rangle \in f^{-1}(\langle V_\alpha \rangle) = \langle f_\alpha^{-1}(V_\alpha) \rangle$. Then $X_\alpha \in f_\alpha^{-1}(V_\alpha)$ for every $\alpha \in A$. Hence, there exists an open set $O_\alpha$ such that $X_\alpha \in O_\alpha \subseteq f_\alpha^{-1}(V_\alpha)$. Hence, $\langle O_\alpha \rangle$ is open in $X$ and $x = \langle O_\alpha \rangle \subseteq \langle f_\alpha^{-1}(V_\alpha) \rangle = f^{-1}(\langle V_\alpha \rangle)$. 

Thus, $f$ is somewhat-continuous on $X$.

\section{Somewhat-connectedness}

This section gives the relationships between somewhat-connectedness, classical connectedness, $\omega_0$-connectedness, and the $\omega$-connectedness. Also, this section presents a characterization of a somewhat-connected space. Denote by $\mathcal{D}$, the topological space $\{0, 1\}$ with the discrete topology.

The proof of the following lemma is standard, hence omitted.

\begin{lemma}
Let $X$ be a topological space and $f_A : X \to \mathcal{D}$ the characteristic function of a subset $A$ of $X$. Then $f_A$ is somewhat-continuous if and only if $A$ is both somewhat-open and somewhat-closed.
\end{lemma}

\begin{theorem}
Let $X$ be a topological space. Then the following statements are equivalent:

(i) $X$ is somewhat-connected.

(ii) The only subsets of $X$ that are both somewhat-open and somewhat-closed are $\emptyset$ and $X$.

(iii) No somewhat-continuous function from $X$ to $\mathcal{D}$ is surjective.
\end{theorem}

\begin{proof}
(i)⇒ (ii): Suppose that $X$ is somewhat-connected and let $G \subseteq X$ which is both somewhat-open and somewhat-closed. Then $X \setminus G$ is also both somewhat-open and somewhat-closed. Moreover, $X = G \cup (X \setminus G)$. Since $X$ is somewhat-connected, either $G = \emptyset$ or $G = X$.

(ii)⇒ (iii): Suppose that $f : X \to \mathcal{D}$ is a somewhat-continuous surjection. Then $f^{-1}(\{0\}) \neq \emptyset, X$. Since $\{0\}$ is open and closed in $\mathcal{D}$, $f^{-1}(\{0\})$ is both somewhat-open and somewhat-closed in $X$. This is a contradiction.

(iii)⇒ (i): If $X = A \cup B$, where $A$ and $B$ are disjoint nonempty somewhat-open sets, then $A$ and $B$ are also somewhat-closed sets. Consider the characteristic function $f_A : X \to \mathcal{D}$ of $A \subseteq X$, which is surjective. By Lemma 4.1, $f_A$ is somewhat-continuous. This is a contradiction. Thus $X$ is somewhat-connected.
\end{proof}

\begin{remark}
Let $X$ be a topological space. If $X$ is somewhat-connected, then $X$ is connected, but not conversely.
\end{remark}

Since every open is somewhat-open, somewhat-connected implies connected. To show that a connected space is not necessarily somewhat-connected, consider $X = \{1, 2, 3, 4\}$ with the topology $T = \{X, \emptyset, \{3\}, \{1, 2\}, \{1, 2, 3\}\}$. Then $X$ is connected but somewhat-disconnected since $A = \{1, 2\}$ and $B = \{3, 4\}$ are somewhat-open sets in $X$ and $A \cup B = X$.

\begin{remark}
Let $X$ be a topological space. Then

(i) somewhat-connected is not necessarily $\omega_0$-connected.

(ii) somewhat-connected is not necessarily $\omega$-connected.
\end{remark}

To verify this, consider $X = \{w, x, y, z\}$ with the topology $T = \{X, \emptyset, \{w\}, \{w, x\}, \{w, z\}, \{w, x, z\}\}$. Then $T_{sw} = \{X, \emptyset, \{w\}, \{w, x\}, \{w, z\}, \{w, x, y\}, \{w, x, z\}, \{w, y, z\}\}$ which means that $X$ is somewhat-connected. But $X$ is $\omega_0$-disconnected and $\omega$-disconnected since $T_{\omega} = T_\omega = P(X)$, where $P(X)$ is the power set of $X$. 

Remark 8 The following diagram holds for any topological space.

\[ \begin{array}{ccc}
\omega_\theta \text{- connected} & \leftrightarrow & \omega \text{- connected} \\
\downarrow & & \uparrow \\
\theta \text{- connected} & \leftrightarrow & \text{connected} \\
& & \\
& \text{somewhat} \text{- connected} & 
\end{array} \]

5. Conclusion

The paper has characterized somewhat-connectedness and described its connection to the other well-known concepts such as the classical connectedness, the \(\omega_\theta\)-connectedness, and the \(\omega\)-connectedness. Moreover, the paper has formulated a necessary and sufficient condition for somewhat-continuity of a function from an arbitrary space into the product space. This particular result is the counterpart of the known characterization of the ordinary continuity of a function into a product space.

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