

## Common fixed point results for graph preserving mappings in parametric $N_b$ -metric spaces

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**Abstract.** In this paper, we discuss the existence and uniqueness of points of coincidence and common fixed points for a pair of graph preserving mappings in parametric  $N_b$ -metric spaces. As some consequences of this study, we obtain several important results in parametric  $b$ -metric spaces, parametric  $S$ -metric spaces and parametric  $A$ -metric spaces. Finally, we provide some illustrative examples to justify the validity of our main result.

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### 1. Introduction

Fixed point theory is an important branch of nonlinear analysis which can be applied to many areas of mathematics and applied sciences such as variational and linear inequalities, control theory, convex optimization, linear algebra, differential equations and mathematical economics. The most celebrated result in this field is the Banach contraction principle [6]. It becomes very famous due to its wide applications. In particular, it is an important tool for solving existence and uniqueness problems in nonlinear functional analysis. Several authors successfully generalized this result in many directions. In last three decades, different types of generalized metric spaces have been developed by different mathematicians. One such generalized metric space is a parametric metric space

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introduced and studied by Hussain et al. [16]. Some other generalized metric spaces are  $b$ -metric space [5], parametric  $b$ -metric space [16], parametric  $S$ -metric space [35] etc.

In 2012, Sedghi et al. [34] introduced the notion of  $S$ -metric space. Afterwards, the definition of  $S$ -metric is generated by extending to  $n$ -tuple by Abbas et al. [1] and called it  $A$ -metric. Recently, Priyobarta et al. [31] introduced the concept of parametric  $A$ -metric space as a generalization of  $A$ -metric space. Very recently, Nihal et al. [36] extended the concept of parametric  $A$ -metric space to parametric  $N_b$ -metric space and studied some fixed point results. After examining the proofs of the results in [36], we noticed that there is something wrong with the proof of the Cauchy sequence in Theorem 3.1 [36]. This leads to subsequent errors in Theorems 4.1 and 5.1 [36]. The detailed reasons are as follows: On page number 950 in [36], the authors used

$$(n-1)ba^k [1 + b^2a + b^4a^2 + \dots] N_{u_0, u_1, t} \leq (n-1) \frac{ba^k}{1 - b^2a} N_{u_0, u_1, t}.$$

This is incorrect unless  $b^2a < 1$ . In this paper, we would like to modify the contractive type condition to achieve their claim (see Corollary 3.4).

In recent investigations, the study of fixed point theory combining a graph is a new development in the domain of contractive type single valued and multi valued theory. In 2005, Echenique [13] studied fixed point theory by using graphs. Later on, Espinola and Kirk [14] applied fixed point results in graph theory. Afterwards, combining fixed point theory and graph theory, a series of articles (see [3, 4, 8, 9, 18, 23–27] and references therein) have been dedicated to the improvement of fixed point theory. Many important results of [1, 11, 21, 22, 28–33, 36] have become the source of motivation for many researchers that do research in fixed point theory. The main purpose of this article is to investigate the existence and uniqueness of points of coincidence and common fixed points for a pair of mappings under various contractive conditions in parametric  $N_b$ -metric spaces. Further, we prove some fixed point theorems for expansive mappings in parametric  $b$ -metric space and parametric  $S$ -metric space.

## 2. Some Basic Concepts

We begin with some basic notations, definitions and results which will be used in the sequel.

**Definition 2.1** [34] Let  $X$  be a nonempty set. An  $S$ -metric on  $X$  is a function  $S : X^3 \rightarrow [0, \infty)$  that satisfies the following conditions, for each  $x, y, z, a \in X$ ,

- (i)  $S(x, y, z) \geq 0$ ,
- (ii)  $S(x, y, z) = 0$  if and only if  $x = y = z$ ,
- (iii)  $S(x, y, z) \leq S(x, x, a) + S(y, y, a) + S(z, z, a)$ .

The pair  $(X, S)$  is called an  $S$ -metric space.

It is to be noted that an  $S$ -metric is not symmetric, in general. The following examples illustrate the above fact.

**Example 2.2** [34] Let  $X = \mathbb{R}^n$  and  $\| \cdot \|$  be a norm on  $X$ . Then

$$S(x, y, z) = \| y + z - 2x \| + \| y - z \|$$

is an  $S$ -metric on  $X$ .

**Example 2.3** [34] Let  $X$  be a nonempty set and  $d$  be an ordinary metric on  $X$ . Then  $S(x, y, z) = d(x, z) + d(y, z)$  is an  $S$ -metric on  $X$ .

**Definition 2.4** [16] Let  $X$  be a nonempty set and  $P : X \times X \times (0, \infty) \rightarrow [0, \infty)$  be a function. Then  $P$  is called a parametric metric on  $X$  if

- (i)  $P(x, y, t) = 0$  if and only if  $x = y$ ,
- (ii)  $P(x, y, t) = P(y, x, t)$ ,
- (iii)  $P(x, y, t) \leq P(x, z, t) + P(z, y, t)$

for each  $x, y, z \in X$  and all  $t > 0$ . The pair  $(X, P)$  is called a parametric metric space.

**Example 2.5** [16] Let  $X$  denote the set of all functions  $f : (0, \infty) \rightarrow \mathbb{R}$ . Define  $P : X \times X \times (0, \infty) \rightarrow [0, \infty)$  by  $P(f, g, t) = |f(t) - g(t)|$  for all  $f, g \in X$  and all  $t > 0$ . Then  $(X, P)$  is a parametric metric space.

**Definition 2.6** [17] Let  $X$  be a nonempty set,  $b \geq 1$  be a real number, and  $P : X \times X \times (0, \infty) \rightarrow [0, \infty)$  be a map satisfying the following conditions:

- (i)  $P(x, y, t) = 0$  if and only if  $x = y$ ,
- (ii)  $P(x, y, t) = P(y, x, t)$ ,
- (iii)  $P(x, y, t) \leq b[P(x, z, t) + P(z, y, t)]$

for each  $x, y, z \in X$  and all  $t > 0$ . Then  $P$  is called a parametric  $b$ -metric on  $X$  and the pair  $(X, P)$  is called a parametric  $b$ -metric space.

**Definition 2.7** [35] Let  $X$  be a nonempty set and  $P_S : X \times X \times X \times (0, \infty) \rightarrow [0, \infty)$  be a function.  $P_S$  is called a parametric  $S$ -metric on  $X$  if

- (PS1)  $P_S(x, y, z, t) = 0$  if and only if  $x = y = z$ ,
- (PS2)  $P_S(x, y, z, t) \leq P_S(x, x, a, t) + P_S(y, y, a, t) + P_S(z, z, a, t)$

for each  $x, y, z, a \in X$  and all  $t > 0$ . The pair  $(X, P_S)$  is called a parametric  $S$ -metric space.

**Example 2.8** [35] Let  $X = \{f \mid f : (0, \infty) \rightarrow \mathbb{R} \text{ be a function}\}$  and the function  $P_S : X \times X \times X \times (0, \infty) \rightarrow [0, \infty)$  be defined by

$$P_S(f, g, h, t) = |f(t) - h(t)| + |g(t) - h(t)|$$

for each  $f, g, h \in X$  and all  $t > 0$ . Then  $P_S$  is a parametric  $S$ -metric and the pair  $(X, P_S)$  is a parametric  $S$ -metric space.

**Lemma 2.9** [35] Let  $(X, P_S)$  be a parametric  $S$ -metric space. Then we have

$$P_S(x, x, y, t) = P_S(y, y, x, t)$$

for each  $x, y \in X$  and all  $t > 0$ .

**Definition 2.10** [1] Let  $X$  be a nonempty set. A function  $A : X^n \rightarrow [0, \infty)$  is called an  $A$ -metric on  $X$  if for any  $x_i, a \in X, i = 1, 2, \dots, n$ , the following conditions hold:

- (A1)  $A(x_1, x_2, x_3, \dots, x_{n-1}, x_n) \geq 0$ ,
- (A2)  $A(x_1, x_2, x_3, \dots, x_{n-1}, x_n) = 0$  if and only if  $x_1 = x_2 = x_3 = \dots = x_n$ ,

$$(A3) \quad A(x_1, x_2, x_3, \dots, x_{n-1}, x_n) \leq \left\{ \begin{array}{l} A(x_1, x_1, x_1, \dots, (x_1)_{n-1}, a) \\ + A(x_2, x_2, x_2, \dots, (x_2)_{n-1}, a) \\ + \dots + A(x_n, x_n, x_n, \dots, (x_n)_{n-1}, a) \end{array} \right\}.$$

The pair  $(X, A)$  is called an  $A$ -metric space.

**Definition 2.11** [31] Let  $X$  be a nonempty set and  $P_A : X^n \times (0, \infty) \rightarrow [0, \infty)$  be a function.  $P_A$  is called a parametric  $A$ -metric on  $X$  if,

$$(PA1) \quad P_A(x_1, x_2, \dots, x_n, t) = 0 \text{ if and only if } x_1 = x_2 = \dots = x_n,$$

$$(PA2) \quad P_A(x_1, x_2, \dots, x_n, t) \leq \left\{ \begin{array}{l} P_A(x_1, x_1, \dots, (x_1)_{n-1}, a, t) \\ + P_A(x_2, x_2, \dots, (x_2)_{n-1}, a, t) \\ + \dots + P_A(x_n, x_n, \dots, (x_n)_{n-1}, a, t) \end{array} \right\}$$

for each  $x_i, a \in X$ ,  $i = 1, 2, 3, \dots, n$  and all  $t > 0$ . The pair  $(X, P_A)$  is called a parametric  $A$ -metric space.

**Example 2.12** [31] Let  $X = \mathbb{R}$  and let the function  $P_A : X^n \times (0, \infty) \rightarrow [0, \infty)$  be defined by

$$P_A(x_1, x_2, \dots, x_n, t) = g(t) (|x_1 - x_2| + |x_2 - x_3| + \dots + |x_n - x_1|),$$

for each  $x_1, x_2, \dots, x_n \in X$  and all  $t > 0$ , where  $g : (0, \infty) \rightarrow (0, \infty)$  is a continuous function. Then  $P_A$  is a parametric  $A$ -metric and the pair  $(X, P_A)$  is a parametric  $A$ -metric space.

**Lemma 2.13** [31] Let  $(X, P_A)$  be a parametric  $A$ -metric space. Then we have

$$P_A(x, x, \dots, x, y, t) = P_A(y, y, \dots, y, x, t),$$

for each  $x, y \in X$  and all  $t > 0$ .

**Definition 2.14** [36] Let  $X$  be a nonempty set,  $b \geq 1$  be a given real number,  $n (\geq 2) \in \mathbb{N}$  and  $N : X^n \times (0, \infty) \rightarrow [0, \infty)$  be a function.  $N$  is called a parametric  $N_b$ -metric on  $X$  if,

$$(N1) \quad N(x_1, x_2, \dots, x_n, t) = 0 \text{ if and only if } x_1 = x_2 = \dots = x_n,$$

$$(N2) \quad N(x_1, x_2, \dots, x_n, t) \leq b \left\{ \begin{array}{l} N(x_1, x_1, \dots, (x_1)_{n-1}, a, t) \\ + N(x_2, x_2, \dots, (x_2)_{n-1}, a, t) \\ + \dots + N(x_n, x_n, \dots, (x_n)_{n-1}, a, t) \end{array} \right\}$$

for each  $x_i, a \in X$ ,  $i = 1, 2, 3, \dots, n$  and all  $t > 0$ . The pair  $(X, N)$  is called a parametric  $N_b$ -metric space. If  $n = 3$ , then  $N$  is called a parametric  $S_b$ -metric on  $X$  and the pair  $(X, N)$  is called a parametric  $S_b$ -metric space.

Throughout the paper, we will denote  $N(x, x, \dots, (x)_{n-1}, y, t)$  by  $N_{x,y,t}$ .

**Example 2.15** Let  $X = \{f \mid f : (0, \infty) \rightarrow \mathbb{R} \text{ be a function}\}$  and let the function

$N : X^3 \times (0, \infty) \rightarrow [0, \infty)$  be defined by

$$N(f, g, h, t) = (|f(t) - g(t)| + |f(t) - h(t)| + |g(t) - h(t)|)^2,$$

for each  $f, g, h \in X$  and all  $t > 0$ . Then  $(X, N)$  is a parametric  $N_b$ -metric space with  $b = 3$  and  $n = 3$ . Because,

$$\begin{aligned} N(f, g, h, t) &= (|f(t) - g(t)| + |f(t) - h(t)| + |g(t) - h(t)|)^2 \\ &\leq 4(|f(t) - \alpha(t)| + |g(t) - \alpha(t)| + |h(t) - \alpha(t)|)^2 \\ &\leq 12(|f(t) - \alpha(t)|^2 + |g(t) - \alpha(t)|^2 + |h(t) - \alpha(t)|^2) \\ &= 3(N_{f,\alpha,t} + N_{g,\alpha,t} + N_{h,\alpha,t}) \end{aligned}$$

for each  $f, g, h, \alpha \in X$  and all  $t > 0$ . But it is not a parametric  $S$ -metric space. In fact,  $(PS2)$  does not hold for  $f(t) = 4, g(t) = 6, h(t) = 8, \alpha(t) = 5$ .

**Lemma 2.16** [36] Let  $(X, N)$  be a parametric  $N_b$ -metric space. Then we have  $N_{x,y,t} \leq bN_{y,x,t}$  and  $N_{y,x,t} \leq bN_{x,y,t}$  for each  $x, y \in X$  and all  $t > 0$ .

**Lemma 2.17** [36] Let  $(X, N)$  be a parametric  $N_b$ -metric space. Then we have

$$N_{x,y,t} \leq b[(n - 1)N_{x,z,t} + N_{y,z,t}] \text{ and } N_{x,y,t} \leq b[(n - 1)N_{x,z,t} + bN_{z,y,t}]$$

for each  $x, y, z \in X$  and all  $t > 0$ .

**Definition 2.18** [36] Let  $(X, N)$  be a parametric  $N_b$ -metric space,  $x \in X$  and  $(x_n)$  be a sequence in  $X$ . Then

- (i)  $(x_n)$  converges to  $x$  if and only if for each  $\epsilon > 0$ , there exists  $n_0 \in \mathbb{N}$  such that  $N_{x_n,x,t} < \epsilon$ , for all  $n \geq n_0$  and all  $t > 0$ , that is,  $\lim_{n \rightarrow \infty} N_{x_n,x,t} = 0$ . We denote this by  $\lim_{n \rightarrow \infty} x_n = x$  or  $x_n \rightarrow x (n \rightarrow \infty)$ .
- (ii)  $(x_n)$  is called a Cauchy sequence if and only if for each  $\epsilon > 0$ , there exists  $n_0 \in \mathbb{N}$  such that  $N_{x_n,x_m,t} < \epsilon$  for all  $n, m \geq n_0$  and all  $t > 0$ , that is,  $\lim_{n,m \rightarrow \infty} N_{x_n,x_m,t} = 0$ .
- (iii)  $(X, N)$  is called complete if and only if every Cauchy sequence in  $X$  is convergent.

**Remark 1** [36] In a parametric  $N_b$ -metric space  $(X, N)$ , the following assertions hold:

- (i) A convergent sequence has a unique limit.
- (ii) Each convergent sequence is Cauchy.

**Lemma 2.19** [36] Let  $(X, N)$  be a parametric  $N_b$ -metric space and  $(u_k), (v_k)$  be two sequences converge to  $u$  and  $v$ , respectively. Then we have

$$\frac{1}{b^2}N_{u,v,t} \leq \liminf_{k \rightarrow \infty} N_{u_k,v_k,t} \leq \limsup_{k \rightarrow \infty} N_{u_k,v_k,t} \leq b^2N_{u,v,t}$$

for all  $t > 0$ . In particular, if  $(v_k)$  is a constant sequence such that  $v_k = v$  for all  $k$ , then we get

$$\frac{1}{b^2}N_{u,v,t} \leq \liminf_{k \rightarrow \infty} N_{u_k,v,t} \leq \limsup_{k \rightarrow \infty} N_{u_k,v,t} \leq b^2N_{u,v,t}$$

for all  $t > 0$ . Also if  $u = v$ , then we have  $\lim_{k \rightarrow \infty} N_{u_k, v, t} = 0$  for all  $t > 0$ .

**Definition 2.20** [2] Let  $T$  and  $S$  be self mappings of a set  $X$ . If  $y = Tx = Sx$  for some  $x$  in  $X$ , then  $x$  is called a coincidence point of  $T$  and  $S$  and  $y$  is called a point of coincidence of  $T$  and  $S$ .

**Definition 2.21** [19] The mappings  $T, S : X \rightarrow X$  are weakly compatible, if for every  $x \in X$ ,  $T(Sx) = S(Tx)$  whenever  $Sx = Tx$ .

**Proposition 2.22** [2] Let  $S$  and  $T$  be weakly compatible selfmaps of a nonempty set  $X$ . If  $S$  and  $T$  have a unique point of coincidence  $y = Sx = Tx$ , then  $y$  is the unique common fixed point of  $S$  and  $T$ .

**Definition 2.23** Let  $(X, N)$  be a parametric  $N_b$ -metric space. A mapping  $f : X \rightarrow X$  is called expansive if there exists a positive number  $k > b^3$  such that  $N_{fx, fy, t} \geq k N_{x, y, t}$  for all  $x, y \in X$  and all  $t > 0$ .

Let  $(X, N)$  be a parametric  $N_b$ -metric space and  $\rho$  be a binary relation over  $X$ . Denote  $R = \rho \cup \rho^{-1}$ . Then  $xRy$  if and only if  $x\rho y$  or  $y\rho x$  for all  $x, y \in X$ .

**Definition 2.24** We say that  $(X, N, R)$  is regular if the following condition holds:

If the sequence  $(x_k)$  in  $X$  and the point  $x \in X$  are such that  $x_k R x_{k+1}$  for all  $k \geq 1$  and  $\lim_{k \rightarrow \infty} N_{x_k, x, t} = 0$ , then there exists a subsequence  $(x_{k_i})$  of  $(x_k)$  such that  $x_{k_i} R x$  for all  $i \geq 1$ .

**Definition 2.25** Let  $(X, N)$  be a parametric  $N_b$ -metric space and  $\rho$  be a binary relation over  $X$ . Then the mapping  $T : X \rightarrow X$  is called comparative if  $T$  maps comparable elements into comparable elements, that is,

$$x, y \in X, xRy \Rightarrow TxRTy.$$

Similarly, for  $f, g : X \rightarrow X$ , we call  $f$  is comparative w.r.t.  $g$  if

$$x, y \in X, gxRgy \Rightarrow fxRfy.$$

We next review some basic notions in graph theory.

Let  $(X, N)$  be a parametric  $N_b$ -metric space. We consider a directed graph  $G$  such that the set of its vertices  $V(G) = X$  and the set of its edges  $E(G)$  contains all the loops, i.e.,  $\Delta \subseteq E(G)$  where  $\Delta = \{(x, x) : x \in X\}$ . We assume that  $G$  has no parallel edges. So we can identify  $G$  with the pair  $(V(G), E(G))$ . By  $G^{-1}$  we denote the graph obtained from  $G$  by reversing the direction of edges i.e.,  $E(G^{-1}) = \{(x, y) \in X \times X : (y, x) \in E(G)\}$ . Let  $\tilde{G}$  denote the undirected graph obtained from  $G$  by ignoring the directions of the edges of  $G$ . Therefore, we consider  $G$  as a directed graph which is symmetric. Thus,  $E(\tilde{G}) = E(G) \cup E(G^{-1})$ . Our graph theory notations and terminology are standard and can be found in all graph theory books, like [7, 12, 15].

**Definition 2.26** Let  $(X, N)$  be a parametric  $N_b$ -metric space and  $G = (V(G), E(G))$  be a graph. Then the mapping  $T : X \rightarrow X$  is called edge preserving if

$$x, y \in X, (x, y) \in E(G) \Rightarrow (Tx, Ty) \in E(G).$$

**Definition 2.27** Let  $(X, N)$  be a parametric  $N_b$ -metric space endowed with a digraph  $G = (V(G), E(G))$  and  $f, g : X \rightarrow X$  be two mappings. Then  $f$  is called edge preserving

w.r.t.  $g$  if  $(gx, gy) \in E(G)$  implies that  $(fx, fy) \in E(G)$  for all  $x, y \in X$ .

**Definition 2.28** Let  $(X, N)$  be a parametric  $N_b$ -metric space and  $f : X \rightarrow X$  be a mapping. Then  $f$  is called continuous if given  $x \in X$  and a sequence  $(x_k)_{k \in \mathbb{N}}$ ,  $x_k \rightarrow x$  implies  $fx_k \rightarrow fx$ .

**Definition 2.29** Let  $(X, N)$  be a parametric  $N_b$ -metric space and  $f, g : X \rightarrow X$  be two mappings. Then  $f$  is called continuous w.r.t.  $g$  if given  $x \in X$  and a sequence  $(gx_k)_{k \in \mathbb{N}}$ ,  $gx_k \rightarrow gx$  implies  $fx_k \rightarrow fx$ .

**Definition 2.30** Let  $(X, N)$  be a parametric  $N_b$ -metric space endowed with a graph  $G = (V(G), E(G))$ . A mapping  $f : X \rightarrow X$  is called  $G$ -continuous if given  $x \in X$  and a sequence  $(x_k)_{k \in \mathbb{N}}$ ,  $x_k \rightarrow x$  and  $(x_k, x_{k+1}) \in E(G)$  for  $k \in \mathbb{N}$  imply  $fx_k \rightarrow fx$ .

**Definition 2.31** Let  $(X, N)$  be a parametric  $N_b$ -metric space endowed with a graph  $G = (V(G), E(G))$  and let  $f, g : X \rightarrow X$  be two mappings. Then  $f$  is called  $G$ -continuous w.r.t.  $g$  if given  $x \in X$  and a sequence  $(gx_k)_{k \in \mathbb{N}}$ ,  $gx_k \rightarrow gx$  and  $(gx_k, gx_{k+1}) \in E(G)$  for  $k \in \mathbb{N}$  imply  $fx_k \rightarrow fx$ .

### 3. Fixed Points in Parametric $N_b$ -Metric Space

In this section, we assume that  $(X, N)$  is a parametric  $N_b$ -metric space and  $G$  is a reflexive digraph such that  $V(G) = X$  and  $G$  has no parallel edges. Let the mappings  $f, g : X \rightarrow X$  be such that  $f(X) \subseteq g(X)$ . Let  $x_0 \in X$  be arbitrary. Since  $f(X) \subseteq g(X)$ , there exists an element  $x_1 \in X$  such that  $gx_1 = fx_0$ . Continuing in this way, we can construct a sequence  $(gx_k)$  such that  $gx_k = fx_{k-1}$ ,  $k = 1, 2, 3, \dots$ .

Before presenting our main result, we state a property of the graph  $G$ , call it property  $(*)$ .

**Property  $(*)$ :** If  $(gx_k)$  is a sequence in  $X$  such that  $gx_k \rightarrow x$  and  $(gx_k, gx_{k+1}) \in E(G)$  for all  $k \geq 1$ , then there exists a subsequence  $(gx_{k_i})$  of  $(gx_k)$  such that  $(gx_{k_i}, x) \in E(G)$  for all  $i \geq 1$ .

Taking  $g = I$ , the above property reduces to property  $(*)'$ :

**Property  $(*)'$ :** If  $(x_k)$  is a sequence in  $X$  such that  $x_k \rightarrow x$  and  $(x_k, x_{k+1}) \in E(G)$  for all  $k \geq 1$ , then there exists a subsequence  $(x_{k_i})$  of  $(x_k)$  such that  $(x_{k_i}, x) \in E(G)$  for all  $i \geq 1$ .

**Theorem 3.1** Let  $(X, N)$  be a parametric  $N_b$ -metric space endowed with a digraph  $G$  and let the mappings  $f, g : X \rightarrow X$  be such that

$$N_{fx,fy,t} \leq h \max \left\{ N_{gx,gy,t}, N_{fx,gx,t}, N_{fy,gy,t}, \frac{N_{gx,fy,t} + N_{gy,fx,t}}{2(n-1)b} \right\} \tag{1}$$

for all  $x, y \in X$  with  $(gx, gy) \in E(\tilde{G})$ , all  $t > 0$  and some  $0 \leq h < \frac{1}{b^3}$ . Suppose that  $f$  is edge preserving w.r.t.  $g$ ,  $f(X) \subseteq g(X)$  and  $f(X)$  or  $g(X)$  is a complete subspace of  $X$ . Assume that at least one of the following conditions holds:

- (i)  $f$  is  $G$ -continuous w.r.t.  $g$ .
- (ii) The graph  $G$  has the property  $(*)$ .

If there exists  $x_0 \in X$  such that  $(gx_0, fx_0) \in E(G)$ , then  $f$  and  $g$  have a point of coincidence in  $g(X)$ . Moreover,  $f$  and  $g$  have a unique point of coincidence in  $g(X)$  if the graph  $G$  has the following property:

(\*\*) If  $x, y$  are points of coincidence of  $f$  and  $g$  in  $g(X)$ , then  $(x, y) \in E(\tilde{G})$ . Furthermore, if  $f$  and  $g$  are weakly compatible, then  $f$  and  $g$  have a unique common fixed point in  $g(X)$ .

**Proof.** Suppose there exists  $x_0 \in X$  such that  $(gx_0, fx_0) \in E(G)$ . Since  $f(X) \subseteq g(X)$ , there exists  $x_1 \in X$  such that  $gx_1 = fx_0$ . So,  $(gx_0, gx_1) \in E(G)$ . As  $f$  is edge preserving w.r.t.  $g$ , it follows that  $(fx_0, fx_1) \in E(G)$ , that is,  $(gx_1, fx_1) \in E(G)$ . Continuing this process, we can construct a sequence  $(x_k)$  in  $X$  such that  $gx_k = fx_{k-1}$ , for all  $k \geq 1$  and  $(gx_k, gx_{k+1}) \in E(G)$  for all  $k \in \mathbb{N} \cup \{0\}$ .

We assume that  $fx_k \neq fx_{k-1}$  for every  $k \in \mathbb{N}$ . If  $fx_k = fx_{k-1}$  for some  $k \in \mathbb{N}$ , then  $gx_k = fx_{k-1} = fx_k$  which shows that  $fx_{k-1}$  is a point of coincidence of  $f$  and  $g$  in  $g(X)$ . By using condition (1) and Lemmas 2.16 and 2.17, we have

$$\begin{aligned}
 N_{fx_k, fx_{k+1}, t} &\leq h \max \left\{ \begin{array}{l} N_{gx_k, gx_{k+1}, t}, N_{fx_k, gx_k, t}, N_{fx_{k+1}, gx_{k+1}, t}, \\ \frac{N_{gx_k, fx_{k+1}, t} + N_{gx_{k+1}, fx_k, t}}{2(n-1)b} \end{array} \right\} \\
 &\leq h \max \left\{ \begin{array}{l} N_{fx_{k-1}, fx_k, t}, N_{fx_k, fx_{k-1}, t}, N_{fx_{k+1}, fx_k, t}, \\ \frac{b N_{fx_{k+1}, fx_{k-1}, t} + N_{fx_k, fx_k, t}}{2(n-1)b} \end{array} \right\} \\
 &\leq h \max \left\{ \begin{array}{l} N_{fx_{k-1}, fx_k, t}, b N_{fx_{k-1}, fx_k, t}, \\ b N_{fx_k, fx_{k+1}, t}, \frac{N_{fx_{k+1}, fx_{k-1}, t}}{2(n-1)} \end{array} \right\} \\
 &\leq h \max \left\{ \begin{array}{l} b N_{fx_{k-1}, fx_k, t}, b N_{fx_k, fx_{k+1}, t}, \\ \frac{(n-1)b N_{fx_{k+1}, fx_k, t} + b N_{fx_{k-1}, fx_k, t}}{2(n-1)} \end{array} \right\} \\
 &\leq h \max \left\{ \begin{array}{l} b N_{fx_{k-1}, fx_k, t}, b^2 N_{fx_k, fx_{k+1}, t}, \\ \frac{(n-1)b^2 N_{fx_k, fx_{k+1}, t} + b(n-1) N_{fx_{k-1}, fx_k, t}}{2(n-1)} \end{array} \right\} \\
 &= h \max \left\{ \begin{array}{l} b N_{fx_{k-1}, fx_k, t}, b^2 N_{fx_k, fx_{k+1}, t}, \\ \frac{b^2 N_{fx_k, fx_{k+1}, t} + b N_{fx_{k-1}, fx_k, t}}{2} \end{array} \right\} \\
 &= h \max \{ b N_{fx_{k-1}, fx_k, t}, b^2 N_{fx_k, fx_{k+1}, t} \}. \tag{2}
 \end{aligned}$$

If  $\max \{ b N_{fx_{k-1}, fx_k, t}, b^2 N_{fx_k, fx_{k+1}, t} \} = b^2 N_{fx_k, fx_{k+1}, t}$ , then condition (2) and  $0 \leq h < \frac{1}{b^3}$  assure that  $N_{fx_k, fx_{k+1}, t} \leq hb^2 N_{fx_k, fx_{k+1}, t} < N_{fx_k, fx_{k+1}, t}$ , which is a contradiction. Thus,  $\max \{ b N_{fx_{k-1}, fx_k, t}, b^2 N_{fx_k, fx_{k+1}, t} \} = b N_{fx_{k-1}, fx_k, t}$ . So, it follows from condition (2) that  $N_{fx_k, fx_{k+1}, t} \leq hb N_{fx_{k-1}, fx_k, t}$  for all  $k \in \mathbb{N}$ . Put  $\alpha = hb$ . Then,  $0 \leq \alpha < \frac{1}{b^2}$ . Therefore,

$$N_{fx_k, fx_{k+1}, t} \leq \alpha N_{fx_{k-1}, fx_k, t}, \text{ for all } k \in \mathbb{N}. \tag{3}$$

By repeated use of condition (3), we get

$$N_{fx_k, fx_{k+1}, t} \leq \alpha^k N_{fx_0, fx_1, t}, \text{ for all } k \geq 0. \tag{4}$$



Therefore,  $\lim_{k \rightarrow \infty} N_{fx_k, fx_{k+1}, t} = 0$ . Now, we show that  $(fx_k)$  is a Cauchy sequence in  $f(X)$ . For  $m, k \in \mathbb{N}$  with  $m > k$  and using conditions (4), (N2), Lemmas 2.16, 2.17, we have

$$\begin{aligned} N_{fx_k, fx_m, t} &\leq (n-1)bN_{fx_k, fx_{k+1}, t} + bN_{fx_m, fx_{k+1}, t} \\ &\leq (n-1)bN_{fx_k, fx_{k+1}, t} + b^2N_{fx_{k+1}, fx_m, t} \\ &\leq (n-1)bN_{fx_k, fx_{k+1}, t} + (n-1)b^3N_{fx_{k+1}, fx_{k+2}, t} + b^3N_{fx_m, fx_{k+2}, t} \\ &\leq (n-1)bN_{fx_k, fx_{k+1}, t} + (n-1)b^3N_{fx_{k+1}, fx_{k+2}, t} + (n-1)b^5N_{fx_{k+2}, fx_{k+3}, t} \\ &\quad + \dots + (n-1)b^{2m-2k-3}N_{fx_{m-2}, fx_{m-1}, t} + b^{2m-2k-2}N_{fx_{m-1}, fx_m, t} \\ &\leq (n-1)b[\alpha^k + b^2\alpha^{k+1} + b^4\alpha^{k+2} + \dots + b^{2(m-k-2)}\alpha^{m-2}]N_{fx_0, fx_1, t} \\ &\quad + (n-1)bb^{2(m-k-1)}\alpha^{m-1}N_{fx_0, fx_1, t} \\ &\leq (n-1)b\alpha^k[1 + (b^2\alpha) + (b^2\alpha)^2 + \dots]N_{fx_0, fx_1, t} \\ &= (n-1)\frac{b\alpha^k}{1 - b^2\alpha}N_{fx_0, fx_1, t} \rightarrow 0 \text{ as } k \rightarrow \infty, \text{ since } b^2\alpha < 1 \end{aligned}$$

Thus,  $\lim_{k, m \rightarrow \infty} N_{fx_k, fx_m, t} = 0$ . This proves that  $(fx_k)$  is a Cauchy sequence in  $f(X)$ . Let  $f(X)$  be a complete subspace of  $X$ . Then there exists  $u \in f(X) \subseteq g(X)$  so that  $fx_k \rightarrow u$  and also  $gx_k \rightarrow u$ . In case,  $g(X)$  is complete, this holds also with  $u \in g(X)$ . Let  $u = gs$  for some  $s \in X$ . Now, we prove that  $u$  is a point of coincidence of  $f$  and  $g$  in  $g(X)$ .

Suppose that condition (i) holds, that is,  $f$  is assumed to be  $G$ -continuous w.r.t.  $g$ . Then,  $gx_k \rightarrow u = gs$  and  $(gx_k, gx_{k+1}) \in E(G)$  imply that  $fx_k \rightarrow fs$ . As the limit of a convergent sequence is unique, it follows that  $u = fs = gs$ . Therefore,  $u$  is a point of coincidence of  $f$  and  $g$  in  $g(X)$ .

Now, suppose that condition (ii) holds, that is, we assume that the graph  $G$  has the property (\*). As  $gx_k \rightarrow u$  and  $(gx_k, gx_{k+1}) \in E(G)$  for all  $k \geq 1$ , by property (\*), there exists a subsequence  $(gx_{k_i})$  of  $(gx_k)$  such that  $(gx_{k_i}, u) \in E(G)$  for all  $i \geq 1$ .

By using condition (1), we obtain

$$\begin{aligned} N_{gs, fs, t} &\leq b[(n-1)N_{gs, fx_{k_i}, t} + N_{fs, fx_{k_i}, t}] \\ &\leq (n-1)b^2N_{fx_{k_i}, gs, t} + bN_{fs, fx_{k_i}, t} \\ &\leq (n-1)b^2N_{fx_{k_i}, gs, t} + bh \max \left\{ \begin{aligned} &N_{gs, gx_{k_i}, t}, N_{fx_{k_i}, gx_{k_i}, t}, N_{fs, gs, t}, \\ &\frac{N_{gs, fx_{k_i}, t} + N_{gx_{k_i}, fs, t}}{2(n-1)b} \end{aligned} \right\} \\ &\leq (n-1)b^2N_{fx_{k_i}, gs, t} + bh \max \left\{ \begin{aligned} &bN_{gx_{k_i}, gs, t}, (n-1)bN_{fx_{k_i}, gs, t} + bN_{gx_{k_i}, gs, t}, \\ &N_{fs, gs, t}, \frac{bN_{fx_{k_i}, gs, t} + (n-1)bN_{gx_{k_i}, gs, t} + bN_{fs, gs, t}}{2b} \end{aligned} \right\}. \end{aligned}$$

Taking the limit as  $i \rightarrow \infty$ , we get

$$N_{gs, fs, t} \leq bh \max \left\{ N_{fs, gs, t}, \frac{1}{2}N_{fs, gs, t} \right\} \leq b^2h N_{gs, fs, t},$$

which gives that  $N_{gs, fs, t} = 0$ , since  $0 \leq hb^2 < \frac{1}{b}$ . Thus,  $gs = fs = u$  and so  $u$  is a point

of coincidence of  $f$  and  $g$  in  $g(X)$ . For uniqueness, let there exist another  $v \in g(X)$  such that  $v = gx = fx$  for some  $x \in X$ . By property (\*\*), it follows that  $(u, v) \in E(\tilde{G})$ . Then,

$$\begin{aligned} N_{u,v,t} &= N_{fs,fx,t} \\ &\leq h \max \left\{ N_{gs,gx,t}, N_{fs,gs,t}, N_{fx,gx,t}, \right. \\ &\quad \left. \frac{N_{gs,fx,t} + N_{gx,fs,t}}{2(n-1)b} \right\} \\ &\leq h \max \left\{ N_{u,v,t}, N_{u,u,t}, N_{v,v,t}, \right. \\ &\quad \left. \frac{N_{u,v,t} + N_{v,u,t}}{2b} \right\} \\ &\leq h \max \left\{ N_{u,v,t}, \frac{b N_{u,v,t} + b N_{u,v,t}}{2b} \right\} \\ &= h \max \{N_{u,v,t}, N_{u,v,t}\} \\ &= h N_{u,v,t}. \end{aligned}$$

This gives that  $N_{u,v,t} = 0$ , since  $0 \leq h < \frac{1}{b^3}$  and hence  $u = v$ . Thus,  $f$  and  $g$  have a unique point of coincidence in  $g(X)$ . If  $f$  and  $g$  are weakly compatible, then by Proposition 2.22,  $f$  and  $g$  have a unique common fixed point in  $g(X)$ . ■

**Corollary 3.2** Let  $(X, N)$  be a parametric  $N_b$ -metric space with the coefficient  $b \geq 1$  and let the mappings  $f, g : X \rightarrow X$  be such that  $f(X) \subseteq g(X)$  and  $f(X)$  or  $g(X)$  is a complete subspace of  $X$  and satisfy the following condition:

$$N_{fx,fy,t} \leq h \max \left\{ N_{gx,gy,t}, N_{fx,gx,t}, N_{fy,gy,t}, \frac{N_{gx,fy,t} + N_{gy,fx,t}}{2(n-1)b} \right\}$$

for all  $x, y \in X$ , all  $t > 0$  and some  $0 \leq h < \frac{1}{b^3}$ . Then  $f$  and  $g$  have a unique point of coincidence in  $g(X)$ . Moreover, if  $f$  and  $g$  are weakly compatible, then  $f$  and  $g$  have a unique common fixed point in  $g(X)$ .

**Proof.** The proof follows from Theorem 3.1 by taking  $G = G_0$ , where  $G_0$  is the complete graph  $(X, X \times X)$ . ■

**Corollary 3.3** Let  $(X, N)$  be a complete parametric  $N_b$ -metric space endowed with a digraph  $G$ . Suppose that the mapping  $f : X \rightarrow X$  is edge preserving and satisfies the following condition

$$N_{fx,fy,t} \leq h \max \left\{ N_{x,y,t}, N_{fx,x,t}, N_{fy,y,t}, \frac{N_{x,fy,t} + N_{y,fx,t}}{2(n-1)b} \right\}$$

for all  $x, y \in X$  with  $(x, y) \in E(\tilde{G})$ , all  $t > 0$  and some  $0 \leq h < \frac{1}{b^3}$ . Assume that at least one of the following conditions holds:

- (i)  $f$  is  $G$ -continuous.
- (ii) The graph  $G$  has the property  $(*)'$ .

If there exists  $x_0 \in X$  such that  $(x_0, fx_0) \in E(G)$ , then  $f$  has a fixed point in  $X$ . Moreover,  $f$  has a unique fixed point in  $X$  if the graph  $G$  has the following property:

$(**')$  If  $x, y$  are fixed points of  $f$  in  $X$ , then  $(x, y) \in E(\tilde{G})$ .

**Proof.** The proof follows from Theorem 3.1 by taking  $g = I$ , the identity map on  $X$ . ■

**Corollary 3.4** Let  $(X, N)$  be a complete parametric  $N_b$ -metric space with the coefficient  $b \geq 1$  and let  $f : X \rightarrow X$  be such that

$$N_{fx, fy, t} \leq h \max\{N_{x, y, t}, N_{fx, xt}, N_{fy, yt}, \frac{N_{x, fy, t} + N_{y, fx, t}}{2(n-1)b}\}$$

for all  $x, y \in X$ , all  $t > 0$  and some  $0 \leq h < \frac{1}{b^3}$ . Then  $f$  has a unique fixed point in  $X$ .

**Proof.** The proof can be obtained from Theorem 3.1 by taking  $G = G_0$  and  $g = I$ . ■

**Corollary 3.5** Let  $(X, N)$  be a complete parametric  $N_b$ -metric space with the coefficient  $b \geq 1$  and let  $g : X \rightarrow X$  be an onto mapping satisfying

$$N_{x, y, t} \leq h \max\{N_{gx, gy, t}, N_{x, gx, t}, N_{y, gy, t}, \frac{N_{gx, y, t} + N_{gy, x, t}}{2(n-1)b}\}$$

for all  $x, y \in X$ , all  $t > 0$  and some  $0 \leq h < \frac{1}{b^3}$ . Then  $g$  has a unique fixed point in  $X$ .

**Proof.** The proof follows from Theorem 3.1 by taking  $G = G_0$  and  $f = I$ . ■

The following result generalizes Theorem 3.1[31] in parametric  $A$ -metric spaces to parametric  $N_b$ -metric spaces.

**Corollary 3.6** Let  $(X, N)$  be a complete parametric  $N_b$ -metric space with the coefficient  $b \geq 1$  and let  $g : X \rightarrow X$  be an onto mapping satisfying

$$N_{gx, gy, t} \geq k N_{x, y, t} \tag{5}$$

for all  $x, y \in X$ , all  $t > 0$  and some  $k > b^3$ . Then  $g$  has a unique fixed point in  $X$ .

**Proof.** Taking  $h = \frac{1}{k}$  and using condition (5), it follows that

$$N_{x, y, t} \leq \frac{1}{k} N_{gx, gy, t} \leq h \max\{N_{gx, gy, t}, N_{x, gx, t}, N_{y, gy, t}, \frac{N_{gx, y, t} + N_{gy, x, t}}{2(n-1)b}\}$$

for all  $x, y \in X$ , all  $t > 0$  and some  $0 \leq h < \frac{1}{b^3}$ . Now the result follows from Corollary 3.5. ■

**Remark 2** Corollary 3.6 ensures the existence of unique fixed point for expansive mappings in parametric  $N_b$ -metric spaces.

**Corollary 3.7** Let  $(X, N)$  be a parametric  $N_b$ -metric space endowed with a binary relation  $\rho$  over  $X$  and let the mappings  $f, g : X \rightarrow X$  satisfy the following condition:

$$N_{fx, fy, t} \leq h \max\{N_{gx, gy, t}, N_{fx, gx, t}, N_{fy, gy, t}, \frac{N_{gx, fy, t} + N_{gy, fx, t}}{2(n-1)b}\}$$

for all  $x, y \in X$  with  $gxRgy$ , where  $R = \rho \cup \rho^{-1}$ , all  $t > 0$  and some  $0 \leq h < \frac{1}{b^3}$ . Suppose that  $f$  is comparative w.r.t.  $g$ ,  $f(X) \subseteq g(X)$  and  $f(X)$  or  $g(X)$  is a complete subspace of  $X$ . Suppose also that the following conditions hold:

- (i)  $(X, N, R)$  is regular,
- (ii) there exists  $x_0 \in X$  such that  $gx_0Rfx_0$ .

Then  $f$  and  $g$  have a point of coincidence in  $g(X)$ . Moreover,  $f$  and  $g$  have a unique point of coincidence in  $g(X)$  if the following property holds:

If  $x, y$  are points of coincidence of  $f$  and  $g$  in  $g(X)$ , then  $xRy$ .

Furthermore, if  $f$  and  $g$  are weakly compatible, then  $f$  and  $g$  have a unique common fixed point in  $g(X)$ .

**Proof.** The proof follows from Theorem 3.1 by taking  $G = (V(G), E(G))$  where  $V(G) = X$ ,  $E(G) = \{(x, y) \in X \times X : xRy\} \cup \Delta$ . ■

**Corollary 3.8** Let  $(X, N)$  be a complete parametric  $N_b$ -metric space endowed with a partial ordering  $\preceq$ . Suppose the mapping  $f : X \rightarrow X$  maps comparable elements into comparable elements and satisfies the following condition:

$$N_{fx, fy, t} \leq h \max \left\{ N_{x, y, t}, N_{fx, x, t}, N_{fy, y, t}, \frac{N_{x, fy, t} + N_{y, fx, t}}{2(n-1)b} \right\}$$

for all  $x, y \in X$  with  $x \preceq y$  or  $y \preceq x$ , all  $t > 0$  and some  $0 \leq h < \frac{1}{b^3}$ . Suppose the triple  $(X, N, \preceq)$  has the following property:

(†) If  $(x_k)$  is a sequence in  $X$  such that  $x_k \rightarrow x$  and  $x_k, x_{k+1}$  are comparable for all  $k \geq 1$ , then there exists a subsequence  $(x_{k_i})$  of  $(x_k)$  such that  $x_{k_i}, x$  are comparable for all  $i \geq 1$ .

If there exists  $x_0 \in X$  such that  $x_0, fx_0$  are comparable, then  $f$  has a fixed point in  $X$ . Moreover,  $f$  has a unique fixed point in  $X$  if the following property holds:

(††) If  $x, y$  are fixed points of  $f$  in  $X$ , then  $x, y$  are comparable.

**Proof.** The proof can be obtained from Theorem 3.1 by taking  $g = I$  and  $G = G_2$ , where the graph  $G_2$  is defined by  $E(G_2) = \{(x, y) \in X \times X : x \preceq y \text{ or } y \preceq x\}$ . ■

**Corollary 3.9** Let  $(X, N)$  be a complete parametric  $N_b$ -metric space with the coefficient  $b \geq 1$  and let  $f : X \rightarrow X$  be such that

$$N_{fx, fy, t} \leq \alpha N_{x, y, t} + \beta N_{fx, x, t} + \gamma N_{fy, y, t} + \delta [N_{x, fy, t} + N_{y, fx, t}] \quad (6)$$

for all  $x, y \in X$ , all  $t > 0$  and  $\alpha, \beta, \gamma, \delta \geq 0$  with  $\alpha + \beta + \gamma + 2(n-1)b\delta < \frac{1}{b^3}$ . Then  $f$  has a unique fixed point in  $X$ .

**Proof.** Condition (6) gives that

$$N_{fx, fy, t} \leq (\alpha + \beta + \gamma + 2(n-1)b\delta) \max \left\{ \begin{array}{l} N_{x, y, t}, N_{fx, x, t}, N_{fy, y, t}, \\ \frac{N_{x, fy, t} + N_{y, fx, t}}{2(n-1)b} \end{array} \right\}$$

for all  $x, y \in X$ , all  $t > 0$ . Taking  $h = \alpha + \beta + \gamma + 2(n-1)b\delta$ , it follows that  $0 \leq h < \frac{1}{b^3}$ . Now applying Corollary 3.4, we obtain the desired result. ■

**Remark 3** We note that several important fixed point results including fixed points for expansive mappings in parametric  $A$ -metric spaces can be obtained by putting  $b = 1$  and choosing different digraphs in Theorem 3.1.

Now we furnish some examples to justify the validity of our main result. The first example shows that the existence and uniqueness of the common fixed point can not follow easily by working in the setting of a usual metric space without any graph. It

should be noticed that Theorem 3.3 [10] can not assure the existence of a common fixed point in the following example.

**Example 3.10** Let  $X = \{1, 2, 3\} \cup [4, \infty)$  and define  $N : X^3 \times (0, \infty) \rightarrow [0, \infty)$  by

$$N(x, y, z, t) = t^3(|x - y| + |x - z| + |y - z|)^2$$

for all  $x, y, z \in X$  and all  $t > 0$ . Then  $(X, N)$  is a complete parametric  $N_b$ -metric space with  $b = 3, n = 3$ . Let  $G$  be a digraph such that  $V(G) = X$  and  $E(G) = \Delta \cup \{(1, 3)\}$ . Let  $f, g : X \rightarrow X$  be defined by

$$fx = \begin{cases} 1, & \text{if } x = 1, 3, \\ 3, & \text{if } x = 2, \\ x^2, & \text{if } x \geq 4 \end{cases} \quad \text{and } gx = \begin{cases} x, & \text{if } x = 1, 2, 3, \\ x + 2, & \text{if } x \geq 4. \end{cases}$$

Then,  $f(X) \subseteq g(X), g(X)$  is a complete subspace of  $X$ . Moreover,  $f$  and  $g$  are weakly compatible. Obviously,  $f$  and  $g$  does not satisfy the contractive condition in Berinde’s [10] meaning. In fact, in the setting of a usual metric space, for  $x = 1, y = 2$ , we have

$$d(fx, fy) = d(1, 3) = 2 > 1 = d(gx, gy).$$

So, Theorem 3.3 [10] can not assure the existence of a common fixed point of  $f$  and  $g$ .

On the other hand,  $f$  is edge preserving w.r.t.  $g$  with  $(gx_0, fx_0) \in E(G)$  for  $x_0 = 1$ . Furthermore, condition (1) holds trivially and it is easy to compute that properties (\*) and (\*\*) hold true. Thus, we have all the conditions of Theorem 3.1 which ensures the existence of a unique common fixed point 1 of  $f$  and  $g$  in  $g(X)$ .

**Remark 4** It is interesting to note that in Example 3.10, the condition

$$N_{fx,fy,t} \leq h \max\{N_{gx,gy,t}, N_{fx,gx,t}, N_{fy,gy,t}, \frac{N_{gx,fy,t} + N_{gy,fx,t}}{4b}\}$$

does not hold for all  $x, y \in X$ , all  $t > 0$  and some  $0 \leq h < \frac{1}{b^3}$ . In fact, for  $x = 1, y = 4$ , we have  $fx = 1, fy = 16, gx = 1, gy = 6$ . Therefore,  $N_{fx,fy,t} = 4t^3 |fx - fy|^2 = 900t^3$  and

$$\max\{N_{gx,gy,t}, N_{fx,gx,t}, N_{fy,gy,t}, \frac{N_{gx,fy,t} + N_{gy,fx,t}}{4b}\} = 4t^3 \max\{25, 0, 100, \frac{125}{6}\} = 400t^3.$$

Now it follows that,

$$\begin{aligned} N_{fx,fy,t} &= 900t^3 = \frac{9}{4}.400t^3 \\ &= \frac{9}{4} \max\{N_{gx,gy,t}, N_{fx,gx,t}, N_{fy,gy,t}, \frac{N_{gx,fy,t} + N_{gy,fx,t}}{4b}\} \\ &> \frac{1}{b^3} \max\{N_{gx,gy,t}, N_{fx,gx,t}, N_{fy,gy,t}, \frac{N_{gx,fy,t} + N_{gy,fx,t}}{4b}\}. \end{aligned}$$

Moreover, we find that for  $x = 1, y = 4$ , we have  $N_{fx, fy, t} = 900t^3 = \frac{225}{9} N_{x, y, t} > N_{x, y, t}$ .

**Example 3.11** Let  $X = \{0, 2, 4\} \cup [5, \infty)$  and define  $N : X^3 \times (0, \infty) \rightarrow [0, \infty)$  by

$$N(x, y, z, t) = t^3(|x - y| + |x - z| + |y - z|)^2$$

for all  $x, y, z \in X$  and all  $t > 0$ . Then  $(X, N)$  is a complete parametric  $N_b$ -metric space with  $b = 3, n = 3$ . Let  $G$  be a digraph such that  $V(G) = X$  and  $E(G) = \Delta \cup \{(2, 4)\}$ . Let  $f, g : X \rightarrow X$  be defined by

$$fx = \begin{cases} 0, & \text{if } x = 0, \\ 4, & \text{if } x = 2, 4, \\ x^2, & \text{if } x \geq 5 \end{cases} \quad \text{and } gx = \begin{cases} x, & \text{if } x = 0, 2, 4, \\ x + 1, & \text{if } x \geq 5. \end{cases}$$

Then,  $f(X) \subseteq g(X)$ ,  $g(X)$  is a complete subspace of  $X$ . Moreover,  $f$  and  $g$  are weakly compatible. Obviously,  $f$  and  $g$  does not satisfy the contractive condition in Berinde's [10] meaning. In fact, in the setting of a usual metric space, for  $x = 0, y = 2$ , we have

$$d(fx, fy) = d(0, 4) = 4 > 2 = d(gx, gy).$$

So, Theorem 3.3 [10] can not assure the existence of a common fixed point of  $f$  and  $g$ .

On the other hand,  $f$  is edge preserving w.r.t.  $g$  with  $(gx_0, fx_0) \in E(G)$  for  $x_0 = 0$ . Furthermore, condition (1) holds trivially and it is easy to verify that property (\*) holds true. Thus, we have all the conditions of Theorem 3.1 except property (\*\*). We find that 0 and 4 are common fixed points of  $f$  and  $g$  in  $g(X)$  and hence they are also points of coincidence of  $f$  and  $g$  in  $g(X)$ , but  $(0, 4) \notin E(\tilde{G})$ . Thus, we can not find unique common fixed point of  $f$  and  $g$  without property (\*\*), although  $f$  and  $g$  are weakly compatible.

#### 4. Fixed Points in Parametric $b$ -Metric Space

In this section, we note that every parametric  $b$ -metric is a parametric  $N_b$ -metric with  $n = 2$ .

**Theorem 4.1** Let  $(X, P)$  be a parametric  $b$ -metric space with the coefficient  $b \geq 1$  and let the mappings  $f, g : X \rightarrow X$  be such that  $f(X) \subseteq g(X)$  and  $f(X)$  or  $g(X)$  is a complete subspace of  $X$  and satisfy the following condition:

$$P(fx, fy, t) \leq h \max \left\{ \begin{array}{l} P(gx, gy, t), P(fx, gx, t), P(fy, gy, t), \\ \frac{P(gx, fy, t) + P(gy, fx, t)}{2b} \end{array} \right\}$$

for all  $x, y \in X$ , all  $t > 0$  and some  $0 \leq h < \frac{1}{b^3}$ . Then  $f$  and  $g$  have a unique point of coincidence in  $g(X)$ . Moreover, if  $f$  and  $g$  are weakly compatible, then  $f$  and  $g$  have a unique common fixed point in  $g(X)$ .

**Proof.** The proof follows from Corollary 3.2 by taking  $n = 2$ . ■

**Corollary 4.2** Let  $(X, P)$  be a complete parametric  $b$ -metric space with the coefficient  $b \geq 1$  and let  $f : X \rightarrow X$  be such that

$$P(fx, fy, t) \leq h \max \left\{ P(x, y, t), P(fx, x, t), P(fy, y, t), \frac{P(x, fy, t) + P(y, fx, t)}{2b} \right\}$$

for all  $x, y \in X$ , all  $t > 0$  and some  $0 \leq h < \frac{1}{b^3}$ . Then  $f$  has a unique fixed point in  $X$ .

**Proof.** The proof can be obtained from Theorem 4.1 by taking  $g = I$ . ■

**Corollary 4.3** Let  $(X, P)$  be a complete parametric  $b$ -metric space with the coefficient  $b \geq 1$  and let  $g : X \rightarrow X$  be an onto mapping satisfying

$$P(x, y, t) \leq h \max \left\{ P(gx, gy, t), P(x, gx, t), P(y, gy, t), \frac{P(gx, y, t) + P(gy, x, t)}{2b} \right\}$$

for all  $x, y \in X$ , all  $t > 0$  and some  $0 \leq h < \frac{1}{b^3}$ . Then  $g$  has a unique fixed point in  $X$ .

**Proof.** The proof follows from Theorem 4.1 by taking  $f = I$ . ■

**Corollary 4.4** Let  $(X, P)$  be a complete parametric  $b$ -metric space with the coefficient  $b \geq 1$  and let  $g : X \rightarrow X$  be an onto mapping satisfying

$$P(gx, gy, t) \geq k P(x, y, t) \tag{7}$$

for all  $x, y \in X$ , all  $t > 0$  and some  $k > b^3$ . Then  $g$  has a unique fixed point in  $X$ .

**Proof.** Taking  $h = \frac{1}{k}$  and using condition (7), it follows that

$$\begin{aligned} P(x, y, t) &\leq \frac{1}{k} P(gx, gy, t) \\ &\leq h \max \left\{ P(gx, gy, t), P(x, gx, t), P(y, gy, t), \frac{P(gx, y, t) + P(gy, x, t)}{2b} \right\} \end{aligned}$$

for all  $x, y \in X$ , all  $t > 0$  and some  $0 \leq h < \frac{1}{b^3}$ . Now the result follows from Corollary 4.3. ■

**Remark 5** Corollary 4.4 ensures the existence of unique fixed point for expansive mappings in parametric  $b$ -metric spaces.

The following result is the analogue of Banach contraction theorem in parametric  $b$ -metric spaces.

**Theorem 4.5** Let  $(X, P)$  be a complete parametric  $b$ -metric space with the coefficient  $b \geq 1$  and let  $f : X \rightarrow X$  be such that

$$P(fx, fy, t) \leq \alpha P(x, y, t) \tag{8}$$

for all  $x, y \in X$ , all  $t > 0$  and some  $0 \leq \alpha < \frac{1}{b^3}$ . Then  $f$  has a unique fixed point in  $X$ .

**Proof.** By using condition (8), it follows that

$$P(fx, fy, t) \leq \alpha P(x, y, t) \\ \leq \alpha \max\{P(x, y, t), P(fx, x, t), P(fy, y, t), \frac{P(x, fy, t) + P(y, fx, t)}{2b}\}$$

for all  $x, y \in X$ , all  $t > 0$  and some  $0 \leq \alpha < \frac{1}{b^3}$ . Now the result follows from Theorem 4.1 by taking  $g = I$ . ■

By an argument similar to that used in Theorem 4.5, we can obtain the following results.

**Theorem 4.6** Let  $(X, P)$  be a complete parametric  $b$ -metric space with the coefficient  $b \geq 1$  and let  $f : X \rightarrow X$  be such that

$$P(fx, fy, t) \leq \alpha P(fx, x, t) + \beta P(fy, y, t)$$

for all  $x, y \in X$ , all  $t > 0$  and  $\alpha, \beta \geq 0$  with  $\alpha + \beta < \frac{1}{b^3}$ . Then  $f$  has a unique fixed point in  $X$ .

**Theorem 4.7** Let  $(X, P)$  be a complete parametric  $b$ -metric space with the coefficient  $b \geq 1$  and let  $f : X \rightarrow X$  be such that

$$P(fx, fy, t) \leq \alpha [P(x, fy, t) + P(y, fx, t)]$$

for all  $x, y \in X$ , all  $t > 0$  and some  $0 \leq 2\alpha < \frac{1}{b^4}$ . Then  $f$  has a unique fixed point in  $X$ .

The following theorem is a generalization of Theorem 3.3[20] which assures the existence of unique fixed point without continuity of the function.

**Theorem 4.8** Let  $(X, P)$  be a complete parametric metric space and let  $f : X \rightarrow X$  be a mapping satisfying the following condition:

$$P(fx, fy, t) \leq \beta [P(fx, x, t) + P(fy, y, t)] + \delta [P(x, fy, t) + P(y, fx, t)]$$

for all  $x, y \in X$ , all  $t > 0$  and  $\beta, \delta \geq 0$  with  $\beta + \delta < \frac{1}{2}$ . Then  $f$  has a unique fixed point in  $X$ .

**Proof.** The proof follows from Corollary 3.9 by taking  $n = 2$ ,  $b = 1$ ,  $\alpha = 0$ ,  $\gamma = \beta$ . ■

**Remark 6** It is worth mentioning that several important fixed point results in parametric metric spaces can be obtained by putting  $n = 2$ ,  $b = 1$  in Theorem 3.1.

## 5. Fixed Points in Parametric $S$ -Metric Space

In this section, we note that every parametric  $S$ -metric is a parametric  $N_b$ -metric with  $n = 3$  and  $b = 1$ .

**Theorem 5.1** Let  $(X, P_S)$  be a parametric  $S$ -metric space and let the mappings  $f, g : X \rightarrow X$  be such that  $f(X) \subseteq g(X)$  and  $f(X)$  or  $g(X)$  is a complete subspace of  $X$  and



satisfy the following condition:

$$P_S(fx, fx, fy, t) \leq h \max \left\{ \begin{array}{l} P_S(gx, gx, gy, t), P_S(fx, fx, gx, t), \\ P_S(fy, fy, gy, t), \frac{P_S(gx, gx, fy, t) + P_S(gy, gy, fx, t)}{4} \end{array} \right\}$$

for all  $x, y \in X$ , all  $t > 0$  and some  $0 \leq h < 1$ . Then  $f$  and  $g$  have a unique point of coincidence in  $g(X)$ . Moreover, if  $f$  and  $g$  are weakly compatible, then  $f$  and  $g$  have a unique common fixed point in  $g(X)$ .

**Proof.** The proof follows from Corollary 3.2 by taking  $n = 3$  and  $b = 1$ . ■

**Corollary 5.2** Let  $(X, P_S)$  be a complete parametric  $S$ -metric space and let  $g : X \rightarrow X$  be an onto mapping satisfying

$$P_S(x, x, y, t) \leq h \max \left\{ \begin{array}{l} P_S(gx, gx, gy, t), P_S(x, x, gx, t), P_S(y, y, gy, t), \\ \frac{P_S(gx, gx, y, t) + P_S(gy, gy, x, t)}{4} \end{array} \right\}$$

for all  $x, y \in X$ , all  $t > 0$  and some  $0 \leq h < 1$ . Then  $g$  has a unique fixed point in  $X$ .

**Proof.** The proof follows from Theorem 5.1 by taking  $f = I$ . ■

The following result gives fixed point for expansive mappings in a parametric  $S$ -metric space. In fact, this is a generalization of Theorem 21[35].

**Corollary 5.3** Let  $(X, P_S)$  be a complete parametric  $S$ -metric space and let  $g : X \rightarrow X$  be an onto mapping satisfying

$$P_S(gx, gx, gy, t) \geq k P_S(x, x, y, t) \tag{9}$$

for all  $x, y \in X$ , all  $t > 0$  and some  $k > 1$ . Then  $g$  has a unique fixed point in  $X$ .

**Proof.** Taking  $h = \frac{1}{k}$  and using condition (9), it follows that

$$\begin{aligned} P_S(x, x, y, t) &\leq \frac{1}{k} P_S(gx, gx, gy, t) \\ &\leq h \max \left\{ \begin{array}{l} P_S(gx, gx, gy, t), P_S(x, x, gx, t), P_S(y, y, gy, t), \\ \frac{P_S(gx, gx, y, t) + P_S(gy, gy, x, t)}{4} \end{array} \right\} \end{aligned}$$

for all  $x, y \in X$ , all  $t > 0$  and some  $0 \leq h < 1$ . Now the result follows from Corollary 5.2. ■

**Corollary 5.4** Let  $(X, P_S)$  be a complete parametric  $S$ -metric space and let  $f : X \rightarrow X$  be such that

$$P_S(fx, fx, fy, t) \leq h \max \left\{ \begin{array}{l} P_S(x, x, y, t), P_S(fx, fx, x, t), \\ P_S(fy, fy, y, t), \frac{P_S(x, x, fy, t) + P_S(y, y, fx, t)}{4} \end{array} \right\}$$

for all  $x, y \in X$ , all  $t > 0$  and some  $0 \leq h < 1$ . Then  $f$  has a unique fixed point in  $X$ .

**Proof.** The proof can be obtained from Theorem 5.1 by taking  $g = I$ . ■

**Corollary 5.5** Let  $(X, P_S)$  be a complete parametric  $S$ -metric space and let  $f : X \rightarrow X$  be such that

$$P_S(fx, fx, fy, t) \leq \alpha P_S(x, x, y, t) + \beta P_S(fx, fx, x, t) + \gamma P_S(fy, fy, y, t) + \delta [P_S(x, x, fy, t) + P_S(y, y, fx, t)] \quad (10)$$

for all  $x, y \in X$ , all  $t > 0$  and  $\alpha, \beta, \gamma, \delta \geq 0$  with  $\alpha + \beta + \gamma + 4\delta < 1$ . Then  $f$  has a unique fixed point in  $X$ .

**Proof.** It follows from condition (10) that

$$P_S(fx, fx, fy, t) \leq (\alpha + \beta + \gamma + 4\delta) \max \left\{ P_S(x, x, y, t), P_S(fx, fx, x, t), P_S(fy, fy, y, t), \frac{P_S(x, x, fy, t) + P_S(y, y, fx, t)}{4} \right\}$$

for all  $x, y \in X$ , all  $t > 0$ . Taking  $h = \alpha + \beta + \gamma + 4\delta$ , it follows that  $0 \leq h < 1$ . Now applying Corollary 5.4, we obtain the desired result. ■

**Theorem 5.6** Let  $(X, P_S)$  be a complete parametric  $S$ -metric space endowed with a binary relation  $\rho$  over  $X$ . Assume that  $f : X \rightarrow X$  is a comparative map which satisfies the following condition:

$$P_S(fx, fx, fy, t) \leq h \max \left\{ P_S(x, x, y, t), P_S(fx, fx, x, t), P_S(fy, fy, y, t), \frac{P_S(x, x, fy, t) + P_S(y, y, fx, t)}{4} \right\}$$

for all  $x, y \in X$  with  $xRy$ , where  $R = \rho \cup \rho^{-1}$ , all  $t > 0$  and some  $0 \leq h < 1$ . Suppose also that the following conditions hold:

- (i)  $(X, P_S, R)$  is regular,
- (ii) there exists  $x_0 \in X$  such that  $x_0 R f x_0$ .

Then  $f$  has a fixed point in  $X$ . Moreover,  $f$  has a unique fixed point in  $X$  if the following property holds:

If  $x, y$  are fixed points of  $f$  in  $X$ , then  $xRy$ .

**Proof.** The proof follows from Corollary 3.7 by taking  $n = 3$ ,  $b = 1$  and  $g = I$ . ■

**Remark 7** It is valuable to note that several important fixed point results in parametric  $S_b$ -metric spaces can be obtained by putting  $n = 3$  and choosing different digraphs  $G$  in Theorem 3.1.

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