Approximate $n$–ideal amenability of module extension Banach algebras

M. Ettefagh$^a$*, S. Etemad$^b$

$^a$Department of Mathematics, Tabriz Branch, Islamic Azad University, Tabriz, Iran. $^b$Department of Mathematics, Azarbaijan Shahid Madani University, Tabriz, Iran.

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Abstract. Let $A$ be a Banach algebra and $X$ be a Banach $A$–bimodule. We study the notion of approximate $n$–ideal amenability for module extension Banach algebras $A \otimes X$. First, we describe the structure of ideals of this kind of algebras and we present the necessary and sufficient conditions for a module extension Banach algebra to be approximately $n$-ideally amenable.

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1. Introduction and preliminaries

Let $A$ be a Banach algebra and $X$ be a Banach $A$–bimodule. Then $X^*$ (the topological dual of $X$) is a Banach $A$–bimodule with the following module actions:

$$(a \cdot x^*, x) = (x^*, x \cdot a) \quad ; \quad (x^* \cdot a, x) = (x^*, a \cdot x),$$

where $a \in A, x \in X$ and $x^* \in X^*$. If $I$ is a two-sided closed ideal in $A$, then $I^*$ also is a Banach $A$–bimodule with the corresponding actions. Also, $I^{(n)}$ the $n$-th dual space of $I$ is a Banach $A$–bimodule for all $n \in \mathbb{N}$.

*Corresponding author.
E-mail address: etefagh@iaut.ac.ir, minaettefagh@gmail.com (M. Ettefagh); sina.etemad@azaruniv.ac.ir, sina.etemad@gmail.com (S. Etemad).
A derivation from $\mathcal{A}$ into $X$ is a linear mapping $D : \mathcal{A} \to X$ satisfying
\[ D(ab) = a \cdot D(b) + D(a) \cdot b, \quad (a, b \in \mathcal{A}). \]

For $x \in X$, the map $\delta_x : \mathcal{A} \to X$ defined by $\delta_x(a) = a \cdot x - x \cdot a$ is a derivation for each $a \in \mathcal{A}$. This kind of derivations are called inner derivations. We denote by $Z^1(\mathcal{A}, X)$, the space of all continuous derivations from $\mathcal{A}$ into $X$ and we denote by $N^1(\mathcal{A}, X)$, the space of all inner derivations from $\mathcal{A}$ into $X$. The quotient space $H^1(\mathcal{A}, X) = Z^1(\mathcal{A}, X)/N^1(\mathcal{A}, X)$ is called the first cohomology group of $\mathcal{A}$ with coefficients in $X$ (see [3, 10]).

The Banach algebra $\mathcal{A}$ is called amenable if every continuous derivation from $\mathcal{A}$ into Banach $\mathcal{A}$–bimodule $X^*$ is inner, i.e. $H^1(\mathcal{A}, X^*) = \{0\}$ for every Banach $\mathcal{A}$–bimodule $X$. This notion was introduced by B. E. Johnson in ([10]). Bade, Curtis and Dales in [1, 3] defined the concept of weak amenability for commutative Banach algebras. Later, Dales, Ghahramani and Gronbaek [4] introduced the concept of $n$–weak amenability of Banach algebras. The Banach algebra $\mathcal{A}$ is $n$–weakly amenable if $H^1(\mathcal{A}, \mathcal{A}^{(n)}) = \{0\}$, $(n \in \mathbb{N})$.

More recently, Eshaghi-Gordji and Yazdanpanah [8] introduced a notion of amenability as follows: $\mathcal{A}$ is ideally amenable [n–ideally amenable $(n \in \mathbb{N})$] if $H^1(\mathcal{A}, I^*) = \{0\}$ $[H^1(\mathcal{A}, I^{(n)}) = \{0\}]$ for every closed two-sided ideal $I$ in $\mathcal{A}$.

In 2008, Monfared [11] discussed another version of amenability named the right character amenability and after that in 2013, Bodaghi et al. [2] turned to the generalized notion of character amenability and the relevant properties. Recently, in [12], Rahimi and Amini studied the concept of amenability modulo an ideal. They proved some results about this issue that inducing the amenability of $l^1(S)$ modulo ideals by certain categories of group congruences on $S$ is equivalent to the amenability of $S$. Along with this work, one can find other newly-published papers on the amenability modulo an ideals of a Banach algebra such as [6, 9, 13, 14].

Ghahramani and Loy introduced a generalized notion of amenability [5]. This new notion was approximate amenability of a Banach algebra. The continuous Derivation $D : \mathcal{A} \to X$ is called approximately inner if there exists a net $(x_\alpha)_\alpha \subseteq X$ such that for every $a \in \mathcal{A}$, $D(a) = \lim_\alpha (a x_\alpha - x_\alpha a)$. Then a Banach algebra $\mathcal{A}$ is approximately amenable if every continuous derivation from $\mathcal{A}$ into $X^*$ is approximately inner for each Banach $\mathcal{A}$–bimodule $X$. Also, $\mathcal{A}$ is approximately $n$–weakly amenable if every continuous derivation from $\mathcal{A}$ into $\mathcal{A}^{(n)}$ is approximately inner $(n \in \mathbb{N})$.

Similarly, we have the notions approximate ideal [n–ideal] amenability for $n \in \mathbb{N}$ in [7, 15].

**Example 1.1** [15] (i) Let $G$ be a locally compact group. Then $M(G)$ is approximately $n – L^1(G)$ amenable.

(ii) Let $G$ be a compact group. Then $L^1(G)^{**}$ is approximately $n – L^1(G)$ amenable.

The direct $l_1$–sum of $\mathcal{A}$ and $X$ is the Banach space $\mathcal{A} \oplus X$ with the following norm
\[ \|(a, x)\| = \|a\| + \|x\|, \quad (a \in \mathcal{A}, x \in X). \]

Also, $\mathcal{A} \oplus X$ is a Banach algebra with the following product
\[ (a_1, x_1) \cdot (a_2, x_2) = (a_1 a_2, x_1 \cdot a_2 + a_1 \cdot x_2). \]

$\mathcal{A} \oplus X$ is called module extension Banach algebra corresponding to $\mathcal{A}$ and $X$ ([16]). On the other hand, we know that $(0 \oplus X)^\perp$ and $(\mathcal{A} \oplus 0)^\perp$ are isometrically isomorph with
$X^*$ and $A^*$ as $A$–bimodules, respectively. So, we have

$$(A \oplus X)^* = (0 \oplus X)^\perp + (A \oplus 0)^\perp$$

where $+$ denotes direct $A$–bimodule $l_\infty$–sum. But, for simplicity, we can write

$$(A \oplus X)^* = A^* + X^*.$$ 

Now, Consider $A^{(n)} + X^{(n)}$ as the underlying space $(A \oplus X)^{(n)}$. Then

$$(A \oplus X)^{(2n)} = A^{(2n)} \oplus_1 X^{(2n)};$$

$$(A \oplus X)^{(2n+1)} = A^{(2n+1)} \oplus_\infty X^{(2n+1)}.$$ 

One can easily prove that $(A \oplus X)^{(n)}$ is a Banach $(A \oplus X)$–bimodule with the following module actions:

(i) If $n$ is odd:

$$(a, x) \cdot (a^{(n)}, x^{(n)}) = (aa^{(n)} + xx^{(n)}, ax^{(n)}),$$

$$(a^{(n)}, x^{(n)}) \cdot (a, x) = (a^{(n)}a + x^{(n)}x, x^{(n)}a);$$

(ii) If $n$ is even:

$$(a, x) \cdot (a^{(n)}, x^{(n)}) = (aa^{(n)}, ax^{(n)} + xa^{(n)}),$$

$$(a^{(n)}, x^{(n)}) \cdot (a, x) = (a^{(n)}a, a^{(n)}x + x^{(n)}a);$$

where $(a, x) \in A \oplus X$ and $(a^{(n)}, x^{(n)}) \in A^{(n)} + X^{(n)} = (A \oplus X)^{(n)}$.

**Remark 1** Let $A$ be a Banach algebra and $X$ be a Banach $A$–bimodule. Then $J$ is a closed ideal in $A \oplus X$ if and only if there is a closed ideal $I$ in $A$ and a closed submodule $Y$ of $X$ such that $J = I \oplus Y$ and that $IX \cup XI \subseteq Y$.

In this paper, we study the approximate $n$–ideal amenability of module extension Banach algebras. Throughout this paper, we consider $I \oplus Y$ as an ideal of Banach algebra $A \oplus X$. Since $(A \oplus X)$–bimodule actions on $(A \oplus X)^{(n)}$ is different whenever $n$ is odd or even, thus approximate $n$–ideal amenability of $A \oplus X$ is investigated in two separate sections 2 and 3.

### 2. approximate $(2n + 1)$–ideal amenability of $A \oplus X$

Throughout this section, $n$ is a non-negative integer. To prove the main theorem of this section, we need the following lemmas.

**Lemma 2.1** Let $T : X \to Y^{(2n+1)}$ be a continuous $A$–bimodule homomorphism. Then $\tilde{T} : A \oplus X \to (I \oplus Y)^{(2n+1)}$ defined by $\tilde{T}(a, x) = (T(x), 0)$ is a continuous derivation. Moreover, $\tilde{T}$ is approximately inner if and only if there exists a net $(F_\alpha)_\alpha \subseteq Y^{(2n+1)}$ such that for every $a \in A$, $\lim_\alpha (aF_\alpha - F_\alpha a) = 0$ and $T(x) = \lim_\alpha (xF_\alpha - F_\alpha x)$ for each $x \in X$. 
Proof. Let \((a_1, x_1)\) and \((a_2, x_2)\) be two arbitrary elements of \(A \oplus X\). We have

\[
\bar{T}((a_1, x_1)(a_2, x_2)) = \bar{T}((a_1a_2, a_1x_2 + x_1a_2)) \\
= (T(a_1x_2 + x_1a_2), 0) \\
= (a_1T(x_2) + T(x_1)a_2, 0).
\]

On the other hand,

\[
(a_1, x_1)\bar{T}((a_2, x_2)) + \bar{T}((a_1, x_1))(a_2, x_2) = (a_1, x_1)(T(x_2), 0) + (T(x_1), 0)(a_2, x_2) \\
= (a_1T(x_2), 0) + (T(x_1)a_2, 0) \\
= (a_1T(x_2) + T(x_1)a_2, 0).
\]

Therefore \(\bar{T}\) is a derivation. Now, Let \(\bar{T}\) be approximately inner. Then there exist nets \((G_\alpha)_{\alpha} \subseteq I^{(2n+1)}\) and \((F_\alpha)_{\alpha} \subseteq Y^{(2n+1)}\) such that

\[
\bar{T}((a, x)) = \lim_{\alpha}((a, x) \cdot (G_\alpha, F_\alpha) - (G_\alpha, F_\alpha) \cdot (a, x)) \\
= \lim_{\alpha}((aG_\alpha + xF_\alpha, aF_\alpha) - (G_\alpha a + F_\alpha x, F_\alpha a)) \\
= \lim_{\alpha}(aG_\alpha + xF_\alpha - G_\alpha a - F_\alpha x, aF_\alpha - F_\alpha a).
\]

Now, for every \(x \in X\), we have

\[
(T(x), 0) = \bar{T}((0, x)) = \lim_{\alpha}[(0, x) \cdot (G_\alpha, F_\alpha) - (G_\alpha, F_\alpha) \cdot (0, x)] \\
= \lim_{\alpha}[(xF_\alpha, 0) - (F_\alpha x, 0)] \\
= [\lim_{\alpha}(xF_\alpha - F_\alpha x), 0] \\
= \lim_{\alpha}(xF_\alpha - F_\alpha x, 0).
\]

Also

\[
(0, 0) = \bar{T}((a, 0)) = \lim_{\alpha}((a, 0) \cdot (G_\alpha, F_\alpha) - (G_\alpha, F_\alpha) \cdot (a, 0)) \\
= \lim_{\alpha}(aG_\alpha - G_\alpha a, aF_\alpha - F_\alpha a).
\]

So, it is clear that for every \(a \in A\), \(\lim_{\alpha}(aF_\alpha - F_\alpha a) = 0\) and for every \(x \in X\), \(T(x) = \lim_{\alpha}(xF_\alpha - F_\alpha x)\).

Conversely, let there exists such a net \((F_\alpha)_{\alpha} \subseteq Y^{(2n+1)}\). Then

\[
\bar{T}((a, x)) = (T(x), 0) = \lim_{\alpha}(xF_\alpha - F_\alpha x, aF_\alpha - F_\alpha a) \\
= \lim_{\alpha}(a, x) \cdot (0, F_\alpha) - (0, F_\alpha) \cdot (a, x).
\]

This shows that \(\bar{T}\) is approximately inner. \(\blacksquare\)

Lemma 2.2 Let \(D : A \to Y^{(2n+1)}\) be a continuous derivation such that for every \(a_1, a_2 \in A\) and \(x_1, x_2 \in X\), \(x_1D(a_2) = D(a_1)x_2\). Then mapping \(\bar{D} : A \oplus X \to (I \oplus Y)^{(2n+1)}\)
defined by \( \tilde{D}((a, x)) = (0, D(a)) \) is a continuous derivation. Moreover

(i) If \( \tilde{D} \) is approximately inner then \( D \) is so.

(ii) If \( D \) is approximately inner then there is a net of continuous derivations \( \tilde{D}_\alpha : \mathcal{A} \oplus X \to (I \oplus Y)^{(2n+1)} \) such that for all \( \alpha \) and for each \( a \in \mathcal{A} \), we have \( \tilde{D}_\alpha((a, 0)) = 0 \) and \( \tilde{D} - \tilde{D}_\alpha \) is inner.

**Proof.** For every \((a_1, x_1), (a_2, x_2) \in \mathcal{A} \oplus X\), we have

\[
\tilde{D}((a_1, x_1)(a_2, x_2)) = \tilde{D}((a_1a_2, a_1x_2 + x_1a_2)) = (0, D(a_1a_2)) = (0, a_1D(a_2) + D(a_1)a_2).
\]

On the other hand,

\[
(a_1, x_1)\tilde{D}((a_2, x_2)) = (a_1, x_1)(0, D(a_2)) = (x_1D(a_2), a_1D(a_2))
\]

and

\[
\tilde{D}((a_1, x_1))(a_2, x_2) = (0, D(a_1))(a_2, x_2) = (D(a_1)x_2, D(a_1)a_2).
\]

It is seen that \( \tilde{D} \) is a derivation.

Now, let \( \tilde{D} \) be approximately inner. Then there are nets \((G_\alpha)_\alpha \subseteq I^{(2n+1)}\) and \((F_\alpha)_\alpha \subseteq Y^{(2n+1)}\) provided that

\[\tilde{D}((a, x)) = \lim_{\alpha}(a, x) \cdot (G_\alpha, F_\alpha) - (G_\alpha, F_\alpha) \cdot (a, x)).\]

But we have

\[
(0, D(a)) = \tilde{D}((a, 0)) = \lim_{\alpha}(a, 0) \cdot (G_\alpha, F_\alpha) - (G_\alpha, F_\alpha) \cdot (a, 0))
\]

\[= \lim_{\alpha}((aG_\alpha, aF_\alpha) - (G_\alpha a, F_\alpha a)) \]

\[= \lim_{\alpha}(aG_\alpha - G_\alpha a, aF_\alpha - F_\alpha a).\]

Hence it follows that \( D(a) = \lim_{\alpha}(aF_\alpha - F_\alpha a) \) for all \( a \in \mathcal{A} \); so \( D \) is approximately inner. This completes the proof of (i).

(ii) Let \( D \) be approximately inner. Then there is a net \((F_\alpha)_\alpha \subseteq Y^{(2n+1)}\) such that for all \( a \in \mathcal{A} \), \( D(a) = \lim_{\alpha}(aF_\alpha - F_\alpha a) \). Suppose that \( T_\alpha : X \to I^{(2n+1)} \) is defined by

\[T_\alpha(x) = F_\alpha x - xF_\alpha, \quad (x \in X).
\]

Also, let \( \tilde{T}_\alpha : \mathcal{A} \oplus X \to (I \oplus Y)^{(2n+1)} \) be defined by

\[\tilde{T}_\alpha((a, x)) = (T_\alpha(x), 0), \quad (a \in \mathcal{A}, x \in X).
\]

Take \( \tilde{D}_\alpha = \tilde{T}_\alpha \). Then for all \( \alpha \) and for each \( a \in \mathcal{A} \) we can write

\[\tilde{D}_\alpha((a, 0)) = \tilde{T}_\alpha((a, 0)) = (T_\alpha(0), 0) = 0.
\]
Thus $\tilde{D}_\alpha((a, 0)) = 0$. On the other hand, we have

$$
(\tilde{D} - \tilde{D}_\alpha)((a, x)) = (\tilde{D} - \tilde{T}_\alpha)((a, x))
$$

$$
= \tilde{D}((a, x)) - \tilde{T}_\alpha((a, x))
$$

$$
= (0, D(a)) - (T_\alpha(x), 0)
$$

$$
= (-T_\alpha(x), D(a))
$$

$$
= (xF_\alpha - F_\alpha x, aF_\alpha - F_\alpha a)
$$

$$
= (a, x) \cdot (0, F_\alpha) - (0, F_\alpha) \cdot (a, x).
$$

Therefore $(\tilde{D} - \tilde{D}_\alpha)$ is inner. 

\[ \square \]

**Lemma 2.3** Suppose that $D : \mathcal{A} \to I^{(2n+1)}$ is a continuous derivation. Then the mapping $\tilde{D} : \mathcal{A} \oplus X \to (I \oplus Y)^{(2n+1)}$ defined by $\tilde{D}((a, x)) = (D(a), 0)$ is a continuous derivation. Moreover, $\tilde{D}$ is approximately inner if and only if $D$ is approximately inner.

**Proof.** Let $(a_1, x_1), (a_2, x_2) \in \mathcal{A} \oplus X$ be two arbitrary elements. We have

$$
\tilde{D}((a_1, x_1) \cdot (a_2, x_2)) = \tilde{D}((a_1 a_2, x_1 a_2 + a_1 x_2)) = (D(a_1 a_2), 0)
$$

$$
= (D(a_1) a_2 + a_1 D(a_2), 0)
$$

$$
= (D(a_1), 0)(a_2, x_2) + (a_1, x_1)(D(a_2), 0)
$$

$$
= \tilde{D}((a_1, x_1))(a_2, x_2) + (a_1, x_1)\tilde{D}((a_2, x_2)).
$$

So, $\tilde{D}$ is a derivation. Now, let $\tilde{D}$ be approximately inner. Then there are nets $(G_\alpha)^{\alpha} \subseteq I^{(2n+1)}$ and $(F_\alpha)^{\alpha} \subseteq Y^{(2n+1)}$ such that

$$
\tilde{D}((a, x)) = \lim_{\alpha}((a, x) \cdot (G_\alpha, F_\alpha) - (G_\alpha, F_\alpha) \cdot (a, x)).
$$

But $\tilde{D}((a, 0)) = (D(a), 0)$. Then it follows that

$$
(D(a), 0) = \tilde{D}((a, 0))
$$

$$
= \lim_{\alpha}((a, 0) \cdot (G_\alpha, F_\alpha) - (G_\alpha, F_\alpha) \cdot (a, 0))
$$

$$
= \lim_{\alpha}(aG_\alpha - G_\alpha a, aF_\alpha - F_\alpha a).
$$

Consequently, $D(a) = \lim_{\alpha}(aG_\alpha - G_\alpha a)$ for $(G_\alpha)^{\alpha} \subseteq I^{(2n+1)}$; i.e. $D$ is approximately inner.

Conversely, we assume that $D$ is approximately inner. Then there is a net $(G_\alpha)^{\alpha} \subseteq I^{(2n+1)}$ such that for all $a \in \mathcal{A}$, $D(a) = \lim_{\alpha}(aG_\alpha - G_\alpha a)$. We can write

$$
\tilde{D}((a, x)) = (D(a), 0) = (\lim_{\alpha}(aG_\alpha - G_\alpha a), 0)
$$

$$
= \lim_{\alpha}(aG_\alpha - G_\alpha a, 0)
$$

$$
= \lim_{\alpha}((a, x) \cdot (G_\alpha, 0) - (G_\alpha, 0) \cdot (a, x)).
$$
By letting \( u_\alpha = (G_\alpha, 0) \subseteq (T \oplus Y)^{(2n+1)} \), we have \( \tilde{D}(a, x) = \lim_{\alpha} (a, x) \cdot \alpha - u_\alpha \cdot (a, x) \)
where \((u_\alpha)_\alpha \subseteq (I \oplus Y)^{(2n+1)}\). Thus \( \tilde{D} \) is approximately inner.

**Lemma 2.4** Let \( T : X \to Y^{(2n+1)} \) be a continuous \( \mathcal{A} \)-bimodule homomorphism satisfying \( x_1 T(x_2) + T(x_1)x_2 = 0 \) for each \( x_1, x_2 \in X \). Then the mapping \( \tilde{T} : \mathcal{A} \oplus X \to (I \oplus Y)^{(2n+1)} \) defined by \( \tilde{T}(a, x) = (0, T(x)) \) is a continuous derivation. Moreover, \( \tilde{T} \) is approximately inner if and only if \( T = 0 \).

**Proof.** First, we show that \( \tilde{T} \) is a derivation. Let \((a_1, x_1), (a_2, x_2) \in \mathcal{A} \oplus X \) be two arbitrary elements. We have

\[
\tilde{T}((a_1, x_1) \cdot (a_2, x_2)) = \tilde{T}((a_1a_2, a_1x_2 + x_1a_2)) \\
= (0, T(a_1x_2 + x_1a_2)) \\
= (0, a_1T(x_2) + T(x_1)a_2).
\]

On the other hand,

\[
\tilde{T}((a_1, x_1)) \cdot (a_2, x_2) = (0, T(x_1))(a_2, x_2) = (T(x_1)x_2, T(x_1)a_2)
\]
and

\[
(a_1, x_1) \cdot \tilde{T}((a_2, x_2)) = (a_1, x_1)(0, T(x_2)) = (x_1T(x_2), a_1T(x_2)).
\]

It follows that \( \tilde{T} \) is a derivation.

Now, let \( \tilde{T} \) is approximately inner. Then there are nets \((G_\alpha)_\alpha \subseteq I^{(2n+1)} \) and \((F_\alpha)_\alpha \subseteq Y^{(2n+1)} \) such that

\[
\tilde{T}((a, x)) = \lim_{\alpha} ((a, x) \cdot (G_\alpha, F_\alpha) - (G_\alpha, F_\alpha) \cdot (a, x)).
\]

Since \( \tilde{T}((a, x)) = \tilde{T}((0, x)) \), thus

\[
(0, T(x)) = \tilde{T}((0, x)) = \lim_{\alpha} ((0, x) \cdot (G_\alpha, F_\alpha) - (G_\alpha, F_\alpha) \cdot (0, x)) \\
= \lim_{\alpha} ((xF_\alpha, 0) - (F_\alpha x, 0)) \\
= \lim_{\alpha} (xF_\alpha - F_\alpha x, 0).
\]

Hence \( T \) is trivial; i.e. \( T = 0 \). Converse is clear.

**Theorem 2.5** Let \( \mathcal{A} \oplus X \) be a module extension Banach algebra \( \mathcal{A} \oplus X \) to be approximately \((2n+1)\)--ideally amenable.

**Theorem 2.5** Let \( \mathcal{A} \oplus X \) be a module extension Banach algebra and \( I \oplus Y \) be an arbitrary closed ideal in \( \mathcal{A} \oplus X \). Then \( \mathcal{A} \oplus X \) is approximately \((2n+1)\)--ideally amenable if and only if the following conditions hold:

(i) \( \mathcal{A} \) is approximately \((2n+1)\)--\( I \)--weakly amenable;
(ii) Every derivation from \( \mathcal{A} \) into \( Y^{(2n+1)} \) is approximately inner;
(iii) For every continuous \( \mathcal{A} \)--bimodule homomorphism \( T : X \to I^{(2n+1)} \), there is net \((F_\alpha)_\alpha \subseteq Y^{(2n+1)} \) such that for each \( a \in \mathcal{A} \), \( \lim_{\alpha} (aF_\alpha - F_\alpha a) = 0 \) and for every \( x \in X \), \( T(x) = \lim_{\alpha} (xF_\alpha - F_\alpha x) \).
(iv) The only continuous $\mathcal{A}$–bimodule homomorphism $T : X \rightarrow Y^{(2n+1)}$ for which $x_1T(x_2) + T(x_1)x_2 = 0$ for all $x_1, x_2 \in X$ in $I^{(2n+1)}$ is $T = 0$.

**Proof.** First, we prove the necessity. Let $\mathcal{A} \oplus X$ be approximately $(2n+1)$–ideally amenable and $I \oplus Y$ be a closed ideal in $\mathcal{A} \oplus X$. Then every continuous derivation from $\mathcal{A} \oplus X$ into $(I \oplus Y)^{(2n+1)}$ is approximately inner. Let $D : \mathcal{A} \rightarrow I^{(2n+1)}$ be a continuous derivation. By Lemma 2.3, the derivation $\tilde{D} : \mathcal{A} \oplus X \rightarrow (I \oplus Y)^{(2n+1)}$ defined by $\tilde{D}((a, x)) = (D(a), 0)$ is approximately inner, thus $D$ is so. That is, $D$ is approximately $(2n+1) - I$–weakly amenable. Therefore condition (i) holds.

Now, suppose that $D : \mathcal{A} \rightarrow Y^{(2n+1)}$ is a continuous derivation. By Lemma 2.2, $D$ is approximately inner and consequently the condition (ii) is complete.

If $T : X \rightarrow I^{(2n+1)}$ is an arbitrary continuous $\mathcal{A}$–bimodule homomorphism then since $\check{T} : \mathcal{A} \oplus X \rightarrow (I \oplus Y)^{(2n+1)}$ defined by $\check{T}((a, x)) = (T(x), 0)$ is approximately inner, by Lemma 2.1, it follows that there exists a net $((F_n)_n) \subseteq Y^{(2n+1)}$ such that for each $a \in \mathcal{A}$, $\lim_n (aF_n - F_n a) = 0$ and for every $x \in X$, we have $T(x) = \lim_n (xF_n - F_n x)$. Thus, condition (iii) follows.

Finally, let $T : X \rightarrow Y^{(2n+1)}$ be a continuous $\mathcal{A}$–bimodule homomorphism satisfying $x_1T(x_2) + T(x_1)x_2 = 0$ in $I^{(2n+1)}$ for each $x_1, x_2 \in X$. Since derivation $\check{T} : \mathcal{A} \oplus X \rightarrow (I \oplus Y)^{(2n+1)}$ defined by $\check{T}((a, x)) = (T(x), 0)$ is approximately inner, thus by Lemma 2.4, we have $T = 0$ and this completes the proof of (iv).

Now, we prove the sufficiency. Let conditions (i)-(iv) hold and $I \oplus Y$ be a closed ideal in $\mathcal{A} \oplus X$. Also, let $D : \mathcal{A} \oplus X \rightarrow (I \oplus Y)^{(2n+1)}$ be a continuous derivation. We show that $D$ is approximately inner. For this, consider the following projection maps:

$$p_1 : (I \oplus Y)^{(2n+1)} \rightarrow I^{(2n+1)} ; \quad p_2 : (I \oplus Y)^{(2n+1)} \rightarrow Y^{(2n+1)}.$$

Also, consider the inclusion maps $k_1 : \mathcal{A} \rightarrow \mathcal{A} \oplus X$ and $k_2 : X \rightarrow \mathcal{A} \oplus X$ by $k_1(a) = (a, 0)$ and $k_2(x) = (0, x)$, respectively. It is clear that $p_1$ and $p_2$ are $\mathcal{A}$–bimodule homomorphisms and $k_1$ is algebra homomorphism. Since $D$ is a continuous derivation, then $D \circ k_1 : \mathcal{A} \rightarrow (I \oplus Y)^{(2n+1)}$ is so. This implies that

$$p_1 \circ D \circ k_1 : \mathcal{A} \rightarrow I^{(2n+1)} ; \quad p_2 \circ D \circ k_1 : \mathcal{A} \rightarrow Y^{(2n+1)}$$

are continuous derivations. In this case, by conditions (i) and (ii), $p_1 \circ D \circ k_1$ and $p_2 \circ D \circ k_1$ are approximately inner. Therefore $D \circ k_1$ is approximately inner. By Lemmas 2.2, 2.3 and 2.4

$$D \circ k_1 = p_1 \circ D \circ k_1 + p_2 \circ D \circ k_1 : \mathcal{A} \oplus X \rightarrow (I \oplus Y)^{(2n+1)}$$

is a continuous derivation. Thus there exists a net of continuous derivations $\tilde{D}_\alpha : \mathcal{A} \oplus X \rightarrow (I \oplus Y)^{(2n+1)}$ such that for every $\alpha$ and for each $a \in \mathcal{A}$, $\tilde{D}_\alpha((a, 0)) = 0$ and $\tilde{D}_\alpha \circ k_1 - \tilde{D}_\alpha$ is inner.

On the other hand, for each $a \in \mathcal{A}$ we have

$$(D - \bar{D} \circ k_1)((a, 0)) = D((a, 0)) - \bar{D} \circ k_1((a, 0))$$

$$= D \circ k_1(a) - D \circ k_1(a) = 0.$$

Take $\tilde{D}_\alpha = D - \bar{D} \circ k_1 + \tilde{D}_\alpha$. Then $\tilde{D}_\alpha : \mathcal{A} \oplus X \rightarrow (I \oplus Y)^{(2n+1)}$ is a continuous derivation satisfying $\tilde{D}_\alpha((a, 0)) = 0$ for each $a \in \mathcal{A}$. 

\[ 244 \text{ M. Ettefagh and S. Etemad / J. Linear. Topological. Algebra. 09(03) (2020) 237-252. } \]
Moreover, for every \(a \in \mathcal{A}\) and \(x \in X\) we have
\[
\hat{D}_\alpha((0, ax)) = \hat{D}_\alpha((a, 0)(0, x)) = (a, 0)\hat{D}_\alpha((0, x)) = a\hat{D}_\alpha((0, x))
\]
and
\[
\hat{D}_\alpha((0, xa)) = \hat{D}_\alpha((0, x)(a, 0)) = \hat{D}_\alpha((0, x)(a, 0)) = \hat{D}_\alpha((0, x))a.
\]

Then \(\hat{D}_\alpha \circ k_2 : X \to (I \oplus Y)^{(2n+1)}\) is a continuous \(\mathcal{A}\)–bimodule homomorphism. By condition (iii), for each \(x\) there is net \((F_\beta^\alpha)_{\beta} \subseteq Y^{(2n+1)}\) such that for each \(a \in \mathcal{A}\),
\[
\lim_{\beta}(aF_\beta^\alpha - F_\beta^\alpha a) = 0\quad \text{and for all } x \in X, p_1 \circ \hat{D}_\alpha \circ k_2(x) = \lim_{\beta}(xF_\beta^\alpha - F_\beta^\alpha x).
\]

Also, for every \(x_1, x_2 \in X\) we can write
\[
([p_2 \circ \hat{D}_\alpha \circ k_2(x_1)]x_2 + x_1[p_2 \circ \hat{D}_\alpha \circ k_2(x_2)], 0) = ([p_2 \circ \hat{D}_\alpha(0, x_1)]x_2, 0)
\]
\[
+ (x_1[p_2 \circ \hat{D}_\alpha(0, x_2)], 0)
\]
\[
= \hat{D}_\alpha((0, x_1))(0, x_2) + (0, x_1)\hat{D}_\alpha((0, x_2))
\]
\[
= \hat{D}_\alpha((0, x_1))(0, x_2) + \hat{D}_\alpha((0, x_1))(0, x_2)
\]
\[
= \hat{D}_\alpha((0, 0)) + \hat{D}_\alpha((0, 0))
\]
\[
= (0, 0).
\]

Consequently, for every \(x_1, x_2 \in X\)
\[
[p_2 \circ \hat{D}_\alpha \circ k_2(x_1)]x_2 + x_1[p_2 \circ \hat{D}_\alpha \circ k_2(x_2)] = 0.
\]

Therefore by the condition (iv), \(p_2 \circ \hat{D}_\alpha \circ k_2 = 0\). Thus, one can write
\[
\hat{D}_\alpha((a, x)) = \hat{D}_\alpha((0, x)) = \hat{D}_\alpha \circ k_2(x)
\]
\[
= (p_1 \circ \hat{D}_\alpha \circ k_2(x), p_2 \circ \hat{D}_\alpha \circ k_2(x))
\]
\[
= \lim_{\beta}(xF_\beta^\alpha - F_\beta^\alpha x, 0)
\]
\[
= \lim_{\beta}((a, x) \cdot (0, F_\beta^\alpha) - (0, F_\beta^\alpha) \cdot (a, x)).
\]

So, \(\hat{D}_\alpha\) is approximately inner. By letting \(D = \hat{D}_\alpha + (\bar{D} \circ k_1 - \hat{D}_\alpha)\), we easily observe that \(D\) is approximately inner. Hence \(\mathcal{A} \oplus X\) is approximately \((2n+1)\)–ideally amenable and proof is complete. \(\blacksquare\)

3. approximate \((2n)\)–ideal amenability of \(\mathcal{A} \oplus X\)

Throughout this section, suppose that \(n \in \mathbb{N}\). First, we prove some lemmas.

Lemma 3.1 Let \(T : X \to (I^{(2n)})\) be a continuous \(\mathcal{A}\)–bimodule homomorphism satisfying \(x_1T(x_2) + T(x_1)x_2 = 0\) for every \(x_1, x_2 \in X\). Then the mapping \(\bar{T} : \mathcal{A} \oplus X \to (I \oplus Y)^{(2n)}\) defined by \(\bar{T}((a, x)) = (T(x), 0)\) is a continuous derivation. Moreover, \(\bar{T}\) is approximately inner if and only if \(T = 0\).
Proof. By Lemma 2.1, it is clear that $\tilde{T}$ is a derivation. Let $\tilde{T}$ be approximately inner. Then there are nets $(G_\alpha)_\alpha \subseteq I^{(2n)}$ and $(F_\alpha)_\alpha \subseteq Y^{(2n)}$ such that for every $(a, x) \in A \oplus X$,

$$\tilde{T}((a, x)) = \lim_\alpha ((a, x) \cdot (G_\alpha, F_\alpha) - (G_\alpha, F_\alpha) \cdot (a, x)).$$

Consequently,

$$(T(x), 0) = \lim_\alpha ((aG_\alpha, aF_\alpha + xG_\alpha) - (G_\alpha a, G_\alpha x + F_\alpha a))$$

$$= \lim_\alpha (aG_\alpha - G_\alpha a, aF_\alpha - F_\alpha a + xG_\alpha - G_\alpha x).$$

But, since $(T(x), 0) = \tilde{T}((0, x))$ so

$$(T(x), 0) = \tilde{T}((0, x)) = \lim_\alpha ((0, x) \cdot (G_\alpha, F_\alpha) - (G_\alpha, F_\alpha) \cdot (0, x))$$

$$= \lim_\alpha ((0, xG_\alpha) - (0, G_\alpha x))$$

$$= \lim_\alpha ((0, xG_\alpha - G_\alpha x)).$$

Therefore, $T(x) = 0$ for each $x \in X$. The converse is clear. \hfill \blacksquare

Lemma 3.2 Let $D : A \to Y^{(2n)}$ is a continuous derivation. Then the mapping $\tilde{D} : A \oplus X \to (I \oplus Y)^{(2n)}$ defined by $\tilde{D}((a, x)) = (0, D(a))$ is a continuous derivation. Moreover, $\tilde{D}$ is approximately inner if and only if $D$ is approximately inner.

Proof. It is clear that $\tilde{D}$ is a derivation. Let $\tilde{D}$ be approximately inner. Then there are nets $(G_\alpha)_\alpha \subseteq I^{(2n)}$ and $(F_\alpha)_\alpha \subseteq Y^{(2n)}$ such that for each $(a, x) \in A \oplus X$, we have

$$\tilde{D}((a, x)) = \lim_\alpha ((a, x) \cdot (G_\alpha, F_\alpha) - (G_\alpha, F_\alpha) \cdot (a, x))$$

$$= \lim_\alpha ((aG_\alpha, aF_\alpha + xG_\alpha) - (G_\alpha a, G_\alpha x + F_\alpha a))$$

$$= \lim_\alpha (aG_\alpha - G_\alpha a, aF_\alpha - F_\alpha a + xG_\alpha - G_\alpha x).$$

But we know that

$$(0, D(a)) = \tilde{D}((a, 0)) = \lim_\alpha ((a, 0) \cdot (G_\alpha, F_\alpha) - (G_\alpha, F_\alpha) \cdot (a, 0))$$

$$= \lim_\alpha (aG_\alpha - G_\alpha a, aF_\alpha - F_\alpha a)$$

and

$$\tilde{D}((0, x)) = \lim_\alpha ((0, x) \cdot (G_\alpha, F_\alpha) - (G_\alpha, F_\alpha) \cdot (0, x))$$

$$= \lim_\alpha (0, xG_\alpha - G_\alpha x).$$

Hence for some $(F_\alpha)_\alpha \subseteq Y^{(2n)}$, we have $D(a) = \lim_\alpha (aF_\alpha - F_\alpha a)$. So $D$ is approximately inner.
Conversely, let $D$ be approximately inner. Then there is net $(F_a)_a \subseteq Y^{(2n)}$ such that for every $a \in A$, $D(a) = \lim_\alpha (aF_a - F_a a)$. Then
\[
\tilde{D}((a, x)) = (0, D(a)) = (0, \lim_\alpha (aF_a - F_a a)) = \lim_\alpha ((a, x) \cdot (0, F_a) - (0, F_a) \cdot (a, x)).
\]
Take $(G_a)_a = (0, F_a) \subseteq (I \oplus Y)^{(2n)}$. Then $D((a, x)) = \lim_\alpha ((a, x) \cdot G_a - G_a \cdot (a, x))$; i.e. $\tilde{D}$ is approximately inner.

**Lemma 3.3** Let $D : A \rightarrow I^{(2n)}$ be a continuous derivation. Then the mapping $\tilde{D} : A \oplus X \rightarrow (I \oplus Y)^{(2n)}$ defined by $\tilde{D}((a, x)) = (D(a), 0)$ is a continuous derivation. Moreover, $\tilde{D}$ is approximately inner if and only if $D$ is approximately inner.

**Proof.** The proof is similar to that of Lemma 2.3. 

**Lemma 3.4** Let $T : X \rightarrow Y^{(2n)}$ be a continuous $A$–bimodule homomorphism. Then the mapping $\tilde{T} : A \oplus X \rightarrow (I \oplus Y)^{(2n)}$ defined by $\tilde{T}((a, x)) = (0, T(x))$ is a continuous derivation. Moreover, $\tilde{T}$ is approximately inner if and only if there exists net $(G_a)_a \subseteq I^{(2n)}$ such that for every $a \in A$, $\lim_\alpha (aG_a - G_a a) = 0$ and for each $x \in X$, $T(x) = \lim_\alpha (xG_a - G_a x)$.

**Proof.** First, we show that $\tilde{T}$ is a derivation. Let $(a_1, x_1), (a_2, x_2) \in A \oplus X$ be two arbitrary elements. We have
\[
\tilde{T}((a_1, x_1) \cdot (a_2, x_2)) = \tilde{T}((a_1a_2, a_1x_2 + x_1a_2)) = (0, T(a_1x_2 + x_1a_2)) = (0, a_1T(x_2) + T(x_1)a_2).
\]
On the other hand,
\[
\tilde{T}((a_1, x_1)) \cdot (a_2, x_2) = (0, T(x_1))(a_2, x_2) = (0, T(x_1)a_2)
\]
and
\[
(a_1, x_1) \cdot \tilde{T}((a_2, x_2)) = (a_1, x_1)(0, T(x_2)) = (0, a_1T(x_2)).
\]
This shows that $\tilde{T}$ is a derivation.

Suppose that $\tilde{T}$ is approximately inner. Then there exist nets $(G_a)_a \subseteq I^{(2n)}$ and $(F_a)_a \subseteq Y^{(2n)}$ such that for every $(a, x) \in A \oplus X$, we have
\[
\tilde{T}((a, x)) = \lim_\alpha ((a, x) \cdot (G_a, F_a) - (G_a, F_a) \cdot (a, x)) = \lim_\alpha (aG_a - G_a a, aF_a - F_a a + xG_a - G_a x).
\]
But
\[
(0, T(x)) = \tilde{T}((0, x)) = \lim_\alpha (0, xG_a - G_a x).
\]
and

\[(0, 0) = \bar{T}((a, 0)) = \lim_{\alpha}(aG_\alpha - G_\alpha a, aF_\alpha - F_\alpha a).\]

This follows that for every \(a \in \mathcal{A}\), \(\lim_{\alpha}(aG_\alpha - G_\alpha a) = 0\) and for every \(x \in X\), \(T(x) = \lim_{\alpha}(xG_\alpha - G_\alpha x)\).

Conversely, suppose that there exists such net \((G_\alpha)_\alpha \subseteq I^{(2n)}\) satisfying \(\lim_{\alpha}(aG_\alpha - G_\alpha a) = 0\) and \(T(x) = \lim_{\alpha}(xG_\alpha - G_\alpha x)\). Then we have

\[
\bar{T}((a, x)) = (0, T(x)) = \lim_{\alpha}(aG_\alpha - G_\alpha a, xG_\alpha - G_\alpha x) = \lim_{\alpha}((a, x) \cdot (G_\alpha, 0) - (G_\alpha, 0) \cdot (a, x)).
\]

By letting \((u_\alpha)_\alpha = (G_\alpha, 0) \subseteq (I \oplus Y)^{(2n)}\), it follows that

\[
\bar{T}((a, x)) = \lim_{\alpha}((a, x) \cdot u_\alpha - u_\alpha \cdot (a, x));
\]

i.e. \(\bar{T}\) is approximately inner. \(\blacksquare\)

Now, we can find the necessary and sufficient conditions for module extension Banach algebra \(\mathcal{A} \oplus X\) to be approximately \((2n)\)–ideally amenable.

**Theorem 3.5** Let \(\mathcal{A} \oplus X\) be a module extension Banach algebra and \(I \oplus Y\) be a closed ideal in \(\mathcal{A} \oplus X\). Then \(\mathcal{A} \oplus X\) is approximately \((2n)\)–ideally amenable if and only if the following conditions hold:

(i) The only continuous derivations \(D : \mathcal{A} \to I^{(2n)}\) for which there is a continuous operator \(T : X \to Y^{(2n)}\) such that \(T(ax) = D(a)x + aT(x)\) and \(T(xa) = xD(a) + T(x)a\) \((a \in \mathcal{A}, x \in X)\) are approximately inner derivations;

(ii) Every continuous derivation from \(\mathcal{A}\) into \(Y^{(2n)}\) is approximately inner;

(iii) The only continuous \(\mathcal{A}\)--bimodule homomorphism \(T : X \to I^{(2n)}\) for which \(x_1T(x_2) + T(x_1)x_2 = 0\) \((x_1, x_2 \in X)\) in \(Y^{(2n)}\) is zero;

(iv) For every continuous \(\mathcal{A}\)--bimodule homomorphism \(T : X \to Y^{(2n)}\), there is net \((G_\alpha)_\alpha \subseteq I^{(2n)}\) such that for each \(a \in \mathcal{A}\), \(\lim_{\alpha}(aG_\alpha - G_\alpha a) = 0\) and for every \(x \in X\), \(T(x) = \lim_{\alpha}(xG_\alpha - G_\alpha x)\).

**Proof.** First, we prove the necessity. Let \(\mathcal{A} \oplus X\) be approximately \((2n)\)–ideally amenable and \(I \oplus Y\) be a closed ideal of it. Then every continuous derivation from \(\mathcal{A} \oplus X\) into \((I \oplus Y)^{(2n)}\) is approximately inner. Let \(D : \mathcal{A} \to I^{(2n)}\) be a continuous derivation including the properties mentioned in the condition (i). We define \(\bar{D} : \mathcal{A} \oplus X \to (I \oplus Y)^{(2n)}\) by

\[
\bar{D}((a, x)) = (D(a), T(x))(a \in \mathcal{A}, x \in X).
\]

Clearly, \(\bar{D}\) is a continuous derivation. Also, \(\bar{D}\) is approximately inner. Thus there are nets \((G_\alpha)_\alpha \subseteq I^{(2n)}\) and \((F_\alpha)_\alpha \subseteq Y^{(2n)}\) such that

\[
\bar{D}((a, x)) = \lim_{\alpha}((a, x) \cdot (G_\alpha, F_\alpha) - (G_\alpha, F_\alpha) \cdot (a, x)).
\]
Consequently

\[(D(a), T(x)) = \lim_\alpha ((aG_\alpha, aF_\alpha + xG_\alpha) - (G_\alpha a, G_\alpha x + F_\alpha a)) \]

\[= \lim_\alpha (aG_\alpha - G_\alpha a, aF_\alpha - F_\alpha a + xG_\alpha - G_\alpha x). \]

Therefore \(D(a) = \lim_\alpha (aG_\alpha - G_\alpha a)\) where \((G_\alpha)_\alpha \subseteq I^{(2n)}\). So \(D\) is approximately inner and the condition (i) holds.

Let \(D : A \to Y^{(2n)}\) be a continuous derivation. Because the continuous derivation \(\bar{D} : A \oplus X \to (I \oplus Y)^{(2n)}\) defined by \(\bar{D}((a, x)) = (0, D(a))\) is approximately inner, so by Lemma 3.2, \(D\) is approximately inner and the condition (ii) is proved.

Now, let \(T : X \to I^{(2n)}\) be a continuous \(A\)-bimodule homomorphism satisfying \(x_1T(x_2) + T(x_1)x_2 = 0\) \((x_1, x_2 \in X)\) in \(Y^{(2n)}\). Since the mapping \(\bar{T} : A \oplus X \to (I \oplus Y)^{(2n)}\) defined by \(\bar{T}((a, x)) = (T(x), 0)\) is approximately inner, so by Lemma 3.1 we have \(T = 0\) and the condition (iii) is completed.

Finally, let \(T : X \to Y^{(2n)}\) be a continuous \(A\)-bimodule homomorphism. Since \(\bar{T} : A \oplus X \to (I \oplus Y)^{(2n)}\) defined by \(\bar{T}((a, x)) = (0, T(x))\) is approximately inner, thus by Lemma 3.4, there is net \((G_\alpha)_\alpha \subseteq I^{(2n)}\) such that for every \(a \in A\), \(\lim_\alpha (aG_\alpha - G_\alpha a) = 0\) and for each \(x \in X\), \(T(x) = \lim_\alpha (xG_\alpha - G_\alpha x)\). Hence condition (iv) is proved.

Now, for proving the sufficiency we assume that the conditions (i)-(iv) hold and that \(I \oplus Y\) is an arbitrary closed ideal in \(A \oplus X\). Also, let \(D : A \oplus X \to (I \oplus Y)^{(2n)}\) be a continuous derivation. We show that \(D\) is approximately inner. For this, consider the following projection maps:

\[p_1 : (I \oplus Y)^{(2n)} \to I^{(2n)}; \quad p_2 : (I \oplus Y)^{(2n)} \to Y^{(2n)}. \]

Also, consider the following inclusion maps:

\[k_1 : A \to A \oplus X; \quad k_2 : X \to A \oplus X. \]

Clearly, \(p_1\) and \(p_2\) are \(A\)-bimodule homomorphisms. Since \(D\) is a continuous derivation, thus \(\bar{D} \circ k_1 : A \to (I \oplus Y)^{(2n)}\) is so. On the other hand,

\[p_1 \circ D \circ k_1 : A \to I^{(2n)}; \quad p_2 \circ D \circ k_1 : A \to Y^{(2n)} \]

are continuous derivations.

**Claim 1:** \(p_1 \circ D \circ k_2 : X \to I^{(2n)}\) is trivial.

Take \(\Delta := p_1 \circ D \circ k_2\). To prove claim 1, it is sufficient to show that \(\Delta\) is a continuous \(A\)-bimodule homomorphism satisfying \(x_1\Delta(x_2) + \Delta(x_1)x_2 = 0\) for every \(x_1, x_2 \in X\) by condition (iii). We have

\[\Delta(ax) = p_1 \circ D \circ k_2(ax) = p_1 \circ D((0, ax))\]

\[= p_1 \circ D((a, 0)(0, x))\]

\[= p_1(D((a, 0))(0, x) + (a, 0)D((0, x)))\]

\[= p_1((a, 0)D((0, x)))\]

\[= p_1(aD \circ k_2(x))\]

\[= a(p_1 \circ D \circ k_2)(x)\]

\[= a\Delta(x). \]
Similarly, \( \Delta(xa) = \Delta(x)a \). So \( \Delta = p_1 \circ D \circ k_2 \) is \( \mathcal{A} \)-bimodule homomorphism. Also, we have

\[
0 = D((0,0)) = D((0,x_1)(0,x_2)) \\
= D((0,x_1)(0,x_2) + (0,x_1)D((0,x_2)) \\
= (0,\Delta(x_1)x_2 + (0,x_1\Delta(x_2)) \\
= (0,x_1\Delta(x_2) + \Delta(x_1)x_2).
\]

Therefore claim 1 holds. Now, we take \( T := p_2 \circ D \circ k_2 : X \to Y^{(2n)} \) and \( D_1 := p_1 \circ D \circ k_1 : \mathcal{A} \to I^{(2n)} \).

**Claim 2:** \( T(ax) = D_1(a)x + aT(x) \) and \( T(xa) = xD_1(a) + T(x)a \) for every \( a \in \mathcal{A}, x \in X \).

To prove the above claim, we have

\[
(0,T(ax)) = (0,p_2 \circ D \circ k_2(ax)) \\
= (0,p_2 \circ D((0,ax))) \\
= D((0,ax)) \\
= D((a,0)(0,x)) \\
= D((a,0))(0,x) + (a,0)D((0,x)) \\
= (0,D_1(a)x) + a(0,T(x)) \\
= (0,D_1(a)x + aT(x)).
\]

Similarly, for each \( a \in \mathcal{A} \) and \( x \in X \) we have

\[
(0,T(xa)) = (0,xD_1(a) + T(x)a).
\]

Hence the claim 2 holds. Consequently, derivation \( D_1 = p_1 \circ D \circ k_1 \) is approximately inner by condition (i).

Now, let there exists net \( (G_\alpha)_\alpha \subseteq I^{(2n)} \) such that for every \( a \in \mathcal{A} \),

\[
D_1(a) = \lim_\alpha (aG_\alpha - G_\alpha a).
\]

Also, let \( T_1 : X \to Y^{(2n)} \) be defined by \( T_1(x) = \lim_\alpha (xG_\alpha - G_\alpha x) \) for each \( x \in X \). Then by claim 2, for \( T - T_1 : X \to Y^{(2n)} \) we have

\[
(T - T_1)(ax) = T(ax) - T_1(ax) \\
= (D_1(a)x + aT(x)) - \lim_\alpha (axG_\alpha - G_\alpha ax) \\
= \lim_\alpha (aG_\alpha - G_\alpha a)x + aT(x) = \lim_\alpha (axG_\alpha - G_\alpha ax) \\
= a \lim_\alpha (G_\alpha x - xG_\alpha) + aT(x) \\
= a(T - T_1)(x)
\]

where \( a \in \mathcal{A} \) and \( x \in X \). Similarly, \( (T - T_1)(xa) = (T - T_1)(x)a \). Therefore \( T - T_1 \) is
a continuous $A$–bimodule homomorphism. Now, by condition (iv), there is net $(v_\beta)_\beta \subseteq I^{(2n)}$ such that for each $a \in A$, $\lim_\beta (av_\beta - v_\beta a) = 0$ and for every $x \in X$, $(T - T_1)(x) = \lim_\beta (xv_\beta - v_\beta x)$. From Lemma 3.4, we know that $T - T_1 : A \oplus X \to (I \oplus Y)^{(2n)}$ defined by

$$T - T_1((a, x)) = (0, (T - T_1)(x))$$

is approximately inner derivation. Since $p_2 \circ D \circ k_1 : A \to Y^{(2n)}$ is a continuous derivation, so by the condition (ii), it is approximately inner. On the other hand, by Lemma 3.2, the mapping $p_2 \circ D \circ k_1 : A \oplus X \to (I \oplus Y)^{(2n)}$ defined by

$$p_2 \circ D \circ k_1((a, x)) = (0, p_2 \circ D \circ k_1(a))$$

is approximately inner derivation. Now, by using claim 1, we have

$$D((a, x)) = (D_1(a), p_2 \circ D \circ k_1(a) + T(x))$$

$$= p_2 \circ D \circ k_1((a, x)) + (T - T_1)((a, x)) + (D_1(a), T(x)).$$

Since every three summands are approximately inner derivations, so $D$ is approximately inner derivation from $A \oplus X$ into $(I \oplus Y)^{(2n)}$. Consequently, $A \oplus X$ is approximately $(2n)$–ideally amenable.

Example 3.6 Let $A^\sharp = A \oplus \mathbb{C}$ be the unitization of a Banach algebra $A$ and $n \in \mathbb{N}$. In this case, we have:

(i) if $A^\sharp$ is approximately $n$–ideally amenable, then $A$ is approximately $n$–ideally amenable.

(ii) if $A$ is approximately $(2n - 1)$–ideally amenable, then $A^\sharp$ is approximately $(2n - 1)$–ideally amenable.

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