Weak separation axioms via almost-ID-sets

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Abstract. The purpose of this paper is to introduce some new classes of almost ideal topological spaces by using the notion of almost-I-open sets and study some of their fundamental properties. We study some low separation axioms in almost ideal topological spaces.

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1. Introduction and preliminaries

Ideals in topological spaces have been considered since 1930 by Kuratowski [6, 7]. The topic has won its importance by the paper of Vaidyanathaswamy [14] in 1945. The ideal concept has been studied in different fields [3, 4, 8–11, 13]. A non-empty collection of subsets of $X$ with hereditary and finite additivity conditions is called an ideal or a dual filter on $X$. Namely a non-empty family $I \subseteq P(X)$, where $P(X)$ is the set of all subsets of $X$, is called an ideal which satisfies (i) $A \in I$ gives $P(A) \subseteq I$ and (ii) $A, B \in I$ implies $A \cup B \in I$. Given a topological space $(X, \tau)$ with an ideal $I$ on $X$, a set operator $(.)^*: P(X) \rightarrow P(X)$, is called a local function [14] of $A$ with respect to $\tau$ and $I$ is defined as follows: for $A \subseteq X, A^*(I, \tau) = \{x \in X : U \cap A \notin I, \text{for every } U \in \tau(x)\}$, where $\tau(x) = \{U \in \tau : x \in U\}$. Kuratowski closure operator $Cl^*(.)$ for a topology

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\( \tau^*(I, \tau) \) is called the \( * \)-topology finer than \( \tau \) is defined by \( Cl^*(A) = A \cup A^*(I, \tau) \) ([14]). Where there is no chance for confusion, we will simply write \( A \) for \( A^*(I, \tau) \) and \( \tau^* \) for \( \tau^*(I, \tau) \). If \( I \) is an ideal on \( X \), then the space \( (X, \tau, I) \) is called an ideal topological space. By a space, we always mean a topological space \( (X, \tau) \) with no separation properties assumed. If \( A \subseteq X \), \( Cl(A) \) and \( Int(A) \) will denote the closure and interior of \( A \) in \( (X, \tau) \), respectively. The almost-\( I \)-open (briefly \( AI \)-open) and almost-\( I \)-closed (briefly \( AI \)-closed) sets are presented by the first author and others in [2]. Utilizing these new concepts the class of almost-\( I \)-continuous functions have been obtained. Both of almost-\( I \)-openness and almost-\( I \)-continuity are considered as generalizations of those \( I \)-openness and \( I \)-continuity of Janković and Hamlett [5] and studied in [1, 12]. A subset \( S \) of an ideal topological space \( (X, \tau, I) \) is said to be \( AI \)-open [2] if \( S \subseteq Cl(Int(S^*)) \), \( X\setminus S \) is called almost-\( I \)-Closed. The collection of all \( AI \)-open sets of \( (X, \tau) \) will be denoted by \( AIO(X, \tau) \). Also, \( AIO(X, x) \) denotes the class of all \( A1 \)-open sets containing \( x \in X \).

2. \( AID \)-sets and its separation axioms

Definition 2.1 A subset \( S \) of an ideal topological \( (X, \tau, I) \) is called almost-\( ID \)-set (\( AID \)) if there exists \( U, V \in AIO(X) \) such that \( U \neq X \) and \( A = U \setminus V \).

Observe that every \( AI \)-open set \( U \) different from \( X \) is an \( AID \) set with \( S = U \) and \( V = \emptyset \).

Definition 2.2 An ideal topological space \( (X, \tau, I) \) is called \( AID_0 \) (resp. \( AIT_0 \)) if for any distinct pair of points \( x \) and \( y \) of \( X \), there exists an \( AID \) set of \( (X, \tau, I) \) containing \( x \) but not \( y \) or \( AID \) set (resp. \( AI \)-open set) of \( (X, \tau, I) \) containing \( y \) but not \( x \).

Definition 2.3 An ideal topological space \( (X, \tau, I) \) is called \( AID_1 \) (resp. \( AIT_1 \)) if for any distinct pair of points \( x \) and \( y \) of \( X \), there exists an \( AID \) set (resp. an \( AI \)-open set) of \( X \) containing \( x \) but not \( y \) and an \( AID \) set (resp. \( AI \)-open set) of \( X \) containing \( y \) but not \( x \).

Definition 2.4 An ideal topological space \( (X, \tau, I) \) is called \( AID_2 \) (resp. \( AIT_2 \)) if for any distinct pair of points \( x \) and \( y \) of \( X \), there exists disjoint \( AID \) sets (resp. an \( AI \)-open set) of \( (X, \tau, I) \) containing \( x \) and \( y \), respectively.

Remark 1

(i) If \((X, \tau, I)\) is \( AIT_i \), then it is \( AID_i \), where \( i = 0, 1, 2 \).

(ii) If \((X, \tau, I)\) is \( AID_i \), then it is \( AID_{i-1} \), where \( i = 1, 2 \).

Example 2.5 Let \( X = \{a, b, c\} \), \( \tau = \{\emptyset, \{a, c\}, X\} \) and \( I = \{\emptyset, \{a\}\} \). Then the ideal topological space \((X, \tau, I)\) is both \( AID_2 \) and \( AID_1 \) but none of \( AIT_2 \) and \( AIT_1 \).

Theorem 2.6 For an ideal topological space \((X, \tau, I)\), the following statements are true:

1. \((X, \tau, I)\) is \( AID_0 \) if and only if it is \( AIT_0 \).
2. \((X, \tau, I)\) is \( AID_1 \) if and only if it is \( AIT_2 \).

Proof.

1. We prove only the necessary condition since the sufficiency is stated in Remark 1 (i).

Necessity. Let \((X, \tau, I)\) be \( AID_0 \). Then for each distinct pair of points \( x, y \in X \), at least one of \( x, y \) say \( x \), belongs to an \( AID \) set \( G \) where \( y \notin G \). Let \( G = U_1 \setminus U_2 \)
such that $U_1 \neq X$ and $U_1, U_2 \in AIO(X)$. Then $x \in U_1$, and for $y \notin G$, we have two cases:
(a) $y \notin U_1$; (b) $y \in U_1$ and $y \in U_2$. In case (a), $x \in U_1$ but $y \notin U_1$; In case (b), $y \in U_2$ but $x \notin U_2$. Hence $X$ is $AID_0$.

(2) **Sufficiency.** Follows directly from Remark 1 (ii).

**Necessity.** Suppose $(X, \tau, I)$ is $AID_1$ space. Then for each distinct pair $x, y \in X$, we have $AID$ sets $G_1, G_2$ such that $x \in G_1$, $y \notin G_1$; $y \in G_2, x \notin G_2$. Let $G_1 = U_1 \setminus U_2$, $G_2 = U_3 \setminus U_4$. From $x \notin G_2$ and $y \in U_4$, we have either $x \notin U_3$ or $(x \in U_3$ and $x \in U_4$). Now we consider two cases:

(i) $x \notin U_3$. By $y \notin G_1$ we have two subcases:
(a) $y \notin U_1$. By $x \in U_1 \setminus U_2$, it follows that $x \in U_1 \setminus (U_2 \cup U_3)$ and by $y \in U_3 \setminus U_4$, we have $y \in U_3 \setminus (U_2 \cup U_4)$. Hence $(U_1 \setminus (U_2 \cup U_3)) \cap (U_3 \setminus (U_2 \cup U_4)) = \emptyset$.
(b) $y \in U_1$ and $y \in U_2$. We have $x \in U_1 \setminus U_2, y \notin U_2$ such that $(U_1 \setminus U_2) \cap U_2 = \emptyset$.
(ii) $x \in U_3$ and $x \in U_4$. We have $y \in U_3 \setminus U_4, x \in U_4$ such that $(U_3 \setminus U_4) \cap U_4 = \emptyset$.

Therefore, $X$ is $AID_2$.

**Definition 2.7** A point $x \in X$ which has only $X$ as the $AI$-neighbourhood is called an $AI$-neat point.

**Theorem 2.8** For an $AID_0$ ideal topological space $(X, \tau, I)$ the following are equivalent:

(i) $(X, \tau, I)$ is $AID_1$.

(ii) $(X, \tau, I)$ has no $AI$-neat point.

**Proof.** (i) $\Rightarrow$ (ii): Since $(X, \tau, I)$ is $AID_1$, then each point of $X$ is contained in an $AID$ set $G = U \setminus V$ and thus in $U$. By definition $U \neq X$. This implies that $x$ is not an $AI$-neat point.

(ii) $\Rightarrow$ (i): If $X$ is $AID_0$, then for each distinct pair of points $x, y \in X$, at least one of them, $x$ (say) has an $AI$-neighbourhood $U$ containing $x$ and not $y$. Thus $U$ which is different from $X$ is an $AID$ set. If $X$ has no an $AI$-neat point then $y$ is not an $AI$-neat point. This means that there exists an $AI$-neighbourhood $V$ of $y$ such that $V \neq X$. Thus $y \in V \setminus U$ but not $x$ and $V \setminus U$ is $AID$. Hence $(X, \tau, I)$ is $AID_1$.

**Definition 2.9** [2] A function $f : (X, \tau, I) \longrightarrow (Y, \sigma, J)$ is said to be $AI$-irresolute if $f^{-1}(V) \in AIO(X)$, for every $V \in AIO(Y)$.

**Theorem 2.10** If $f : (X, \tau, I) \longrightarrow (Y, \sigma, J)$ is an $AI$-irresolute surjective function and $E$ is an $AID$ set in $(Y, \sigma, J)$, then the inverse image of $E$ is an $AID$ set in $(X, \tau, I)$.

**Proof.** Let $E$ be an $AID$ set in $(Y, \sigma, J)$. Then, there are $AI$-open sets $U_1$ and $U_2$ in $(Y, \sigma, J)$ such that $E = U_1 \setminus U_2$ and $U \neq Y$. By the $AI$-irresoluteness of $f$, $f^{-1}(U_1)$ and $f^{-1}(U_2)$ are $AI$-open in $(X, \tau, I)$. Since $U_1 \neq Y$, we have $f^{-1}(U_1) \neq X$. Hence $f^{-1}(E) = f^{-1}(U_1 \setminus U_2) \subset f^{-1}(U_1) \setminus f^{-1}(U_2)$ is an $AID$ set.

**Theorem 2.11** If $(Y, \sigma, J)$ is $AID_1$ and $f : (X, \tau, I) \longrightarrow (Y, \sigma, J)$ is $AI$-irresolute and bijective, then $(X, \tau, I)$ is $AID_1$.

**Proof.** Suppose that $Y$ is an $AID_1$ space. Let $x$ and $y$ be any pair of distinct points in $X$. Since $f$ is injective and $Y$ is $AID_1$, there exist $AID$ sets $G_x$ and $G_y$ of $Y$ containing $f(x)$ and $f(y)$, respectively, such that $f(y) \notin G_x$ and $f(x) \notin G_y$. By Theorem 2.10, $f^{-1}(G_x)$ and $f^{-1}(G_y)$ are $AID$ sets in $(X, \tau, I)$ containing $x$ and $y$, respectively. This implies that $(X, \tau, I)$ is an $AID_1$ space.
Theorem 2.12 An ideal topological space \((X, \tau, I)\) is \(AID_1\) if and only if for each pair of distinct points \(x, y \in X\), there exists an \(AI\)-irresolute surjective function \(f : (X, \tau, I) \rightarrow (Y, \sigma, J)\), where \((Y, \sigma, J)\) is an \(AID_1\) space such that \(f(x) \neq f(y)\) are distinct.

Proof. Necessity. For every pair of distinct points of \(X\), it suffices to take the identity function on \(X\).

Sufficiency. Let \(x\) and \(y\) be any pair of distinct points of \(X\). By hypothesis, there exists an \(AI\)-irresolute, surjective function \(f\) from an ideal topological space \((X, \tau, I)\) onto an \(AID_1\) space \((Y, \sigma, J)\) such that \(f(x) \neq f(y)\). Therefore, there exist disjoint \(AID\) sets \(G_x\) and \(G_y\) in \(Y\) such that \(f(x) \in G_x\) and \(f(y) \in G_y\). Since \(f\) is \(AI\)-irresolute and surjective, by Theorem 2.10, \(f^{-1}(G_x)\) and \(f^{-1}(G_y)\) are disjoint \(AID\) sets in \(X\) containing \(x\) and \(y\), respectively. Hence the space \((X, \tau, I)\) is an \(AID_1\) space. ■

Problem 1. Find an \(AID_0\) space which is not \(AIT_0\).

Problem 2. Find an ideal topological space \(AID_{i-1}\) which is not \(AID_i\), where \(i = 1, 2\).

3. Conclusion and further works

We hope this paper is just a beginning of a new structure. It will inspire many to contribute to the cultivation of ideal topology in the field of mathematical structure of approximations. As further works, using the results in this paper, we will further study the following research areas: extended almost-\(I\)-open sets structures, ideals topological data analysis, geographical model, big data analysis and statistics analysis.

References