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# A new implicit iteration process for approximating common fixed points of $\alpha$-demicontraction semigroup

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**Abstract.** It is our purpose in this paper to introduce the concept of $\alpha$-demicontractive semigroup. Also, we construct a new implicit iterative scheme for approximating the common fixed points of $\alpha$-demicontractive semigroup. We prove strong convergence of our new iterative scheme to the common fixed points of $\alpha$-demicontractive semigroup in Banach spaces. Our result is an improvement and generalization of several well known results in the existing literature.



**Keywords:** Fixed point, Banach space, implicit iteration process, $\alpha$-demicontraction semigroup, strong convergence.

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## 1. Introduction

Throughout this paper, we assume that $B$ is a real Banach space with norm $\|\cdot\|$, $B^*$ the dual space of $B$, $\langle\cdot,\cdot\rangle$ the duality between $B$ and $B^*$ and $\Upsilon$ a nonempty closed convex subset of $E$. Let $\Re$ denote the set of nonnegative real numbers and $\mathbb{N}$ denote the natural number set. The mapping $J : B \to 2^{B^*}$ with

$$J(w) = \{f^* \in B^* : \langle w, f^*\rangle = \|w\|^2, \|f^*\| = \|w\|\}, \text{ for all } w \in B, \tag{1}$$

is called the normalized duality mapping. In the sequel, we shall use $j$ to denote the single-valued duality mapping. Let $S : \Upsilon \to \Upsilon$ be a nonlinear mapping. $F(S)$ denote the

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set of fixed point of $S$, i.e, $F(S) = \{w \in \Upsilon : Sw = w\}$. We use "$\to$" to stand for strong convergence. Firstly, we recall the following definitions:

**Definition 1.1** A mapping $S : \Upsilon \to \Upsilon$ is said to be

- nonexpansive if

$$\|Sw - Sp\| \leqslant \|w - p\|, \text{ for all } w, p \in \Upsilon; \tag{2}$$

- strictly pseudocontractive if there exists a constant $\lambda \in (0, 1)$ and $j(w - p) \in J(w - p)$ such that

$$\langle Sw - Sp, j(w - p)\rangle \leqslant \|w - p\|^2 - \lambda\|(I - S)w - (I - S)p\|^2; \tag{3}$$

for all $w, p \in \Upsilon$;
- demicontractive if $F(S) \neq \emptyset$, there exists a constant $\lambda \in (0, 1)$ and $j(w - z) \in J(w - z)$ such that

$$\langle Sw - z, j(w - z)\rangle \leqslant \|w - z\|^2 - \lambda\|w - Sw\|^2, \tag{4}$$

for all $w \in \Upsilon$ and $z \in F(S)$;
- $\alpha$-demicontractive if $F(S) \neq \emptyset$, there exists a constant $\lambda \in (0, 1)$ and $j(w - \alpha z) \in J(w - \alpha z)$ such that

$$\langle Sw - \alpha z, j(w - \alpha z)\rangle \leqslant \|w - \alpha z\|^2 - \lambda\|w - S\|^2, \tag{5}$$

for some $\alpha \geqslant 1$, for all $w \in \Upsilon$ and $z \in F(S)$.

This class of mapping was introduced by Maruster and Maruster [6] in 2011. In [6], Maruster and Maruster proved that the class of $\alpha$-demicontractive mapping is more general than the class of demicontractive mapping with an example, i.e., an example of an $\alpha$-demicontractive mapping with $\alpha > 1$ which is not a demicontractive mapping was given. Clearly, every demicontractive mapping is an $\alpha$-demicontractive with $\alpha = 1$. This implies that the class of demicontractive mappings is a proper subclass of the class of $\alpha$-demicontractive mappings.

***Remark 1*** *[6] Obviously, if $S$ is an $\alpha$-demicontractive mapping then $\alpha z$ is a fixed point of $S$, i.e., $\alpha z \in F(S)$, $\forall z \in F(S)$ and for some $\alpha \geqslant 1$ such that $\alpha z$ remains in the domain $D(S)$ of $S$. For more about the properties of "$\alpha$", the reader can see Maruster and Maruster [6], Osilike and Onah [11].*

**Definition 1.2** A one parameter family $\Im = \{S(s) : s \geqslant 0\}$ of self mappings of $\Upsilon$ is said to be nonexpansive semigroup; if the following conditions are satisfied:

(i) $S(s_1 + s_2)w = S(s_1)S(s_2)w$, for any $s_1, s_2 \in \Re^+$ and $w \in \Upsilon$;
(ii) $S(0)w = w$, for each $w \in \Upsilon$;
(iii) for each $w \in \Upsilon$, $s \mapsto S(s)w$ is continuous;
(iv) for any $s \geqslant 0$, $S(s)$ is nonexpansive on $\Upsilon$, that is for any $w, p \in \Upsilon$,

$$\|S(s)w - S(s)p\| \leqslant \|w - p\|, \text{ for any } s \geqslant 0. \tag{6}$$

If the family $\Im = \{S(s) : s \geqslant 0\}$ satisfies conditions (i)–(iii), then it is said to be:

(a) Lipschitzian semigroup, if there exists a bounded measurable function $L : [0,\infty) \to [0,\infty)$ such that, for any $w, p \in \Upsilon$ and $s \geqslant 0$,

$$\|S(s)w - S(s)p\| \leqslant L(s)\|w - p\|; \tag{7}$$

(b) strictly pseudocontractive semigroup, if there exists a bounded function $\lambda : [0,\infty) \to [0,\infty)$ and for any given $w, p \in \Upsilon$, there exists $j(w-p) \in J(w-p)$ such that

$$\langle S(s)w - S(s)p, j(w-p)\rangle \leqslant \|w-p\|^2 - \lambda(s)\|(I - S(s))w - (I - S(s))p\|^2, \tag{8}$$

for any $s \geqslant 0$;

(c) demicontractive semigroup, if $\bigcap_{s\geqslant 0} F(S(s)) \neq \emptyset$ for all $s \geqslant 0$ and $z \in \bigcap_{s\geqslant 0} F(S(s))$, there exists a bounded function $\lambda : [0,\infty) \to [0,\infty)$ and for any given $w \in \Upsilon$, there exists $j(w-z) \in J(w-z)$ such that

$$\langle S(s)w - z, j(w-z)\rangle \leqslant \|w-z\|^2 - \lambda(s)\|w - S(s)w\|^2. \tag{9}$$

In this paper, we introduce the following semigroup.

**Definition 1.3** A one parameter family $\Im = \{S(s) : s \geqslant 0\}$ of self mapping of $\Upsilon$ satisfying condition (i)–(iii) is said to be $\alpha$-demicontractive semigroup if $\bigcap_{s\geqslant 0} F(S(s)) \neq \emptyset$ for all $s \geqslant 0$, there exists a bounded function $\lambda : [0,\infty) \to [0,\infty)$, for some $\alpha \geqslant 1$, for any $w \in \Upsilon$ and $z \in \bigcap_{s\geqslant 0} F(S(s))$, there exists $j(w-\alpha z) \in J(w-\alpha z)$ such that

$$\langle S(s)w - \alpha z, j(w-\alpha z)\rangle \leqslant \|w-\alpha z\|^2 - \lambda(s)\|w - S(s)w\|^2. \tag{10}$$

***Remark 2*** *Every demicontraction semigroup is an $\alpha$-demicontraction semigroup with $\alpha = 1$. Every strictly pseudocontractive semigroup with a nonempty fixed point is a demicontracive semigroup and $(1+\lambda(s))/\lambda(s)$ Lipschitzian.*

On the other hand, the convergence problems of semigroup has been considered by many authors in the past three decades. Several implicit and explicit schemes have been introduced and studied by many researchers in nonlinear analysis for approximating the common fixed points of nonexpansive seemigroup, strictly pseudocontractive semigroup and demicontractive semigroup (see for example, [1–3, 5, 7–10, 13–15, 18–22] and the references there in).

In 1998, Shioji and Takahashi [13] first introduced and rigorously studied a Halpern-type implicit iterative scheme for approximating the common fixed point of a family of asymptotically nonexpansive semigroup in a Hilbert space.

In [14], Suzuki introduced the following implicit iteration process for finding the common fixed point of nonexpansive semigroup in a Hilbert space:

$$\begin{cases} w_0 \in \Upsilon, \\ w_n = m_n u + (1-m_n)S(s_n)w_n, \end{cases} \quad \text{for all } n \geqslant 1. \tag{11}$$

He proved the strong convergence of (11) to a common fixed point of nonexpansive semigroup by imposing some appropriate conditions on $\{m_n\}$ and $\{s_n\}$.

In 2005, Xu [19] extended the result of Suzuki [14] from Hilbert space to the more general uniformly convex Banach space with a weakly continuous duality mapping.

In 2005, Aleyner and Reich [1] first introduced and studied the following explicit Halpern-type iteration process for approximating the common fixed point of a family $\{S(s) : s \geqslant 0\}$ of nonexpansive semigroup in a reflexive Banach space with a uniformly Gâteaux differentiable norm:

$$\begin{cases} w_0 \in \Upsilon, \\ w_{n+1} = m_n u + (1 - m_n) S(s_n) w_n, \end{cases} \quad \text{for all } n \geqslant 1. \tag{12}$$

In 2007, Zhang et al. [22] introduced the following two steps iteration process for approximating the common fixed point of nonexpansive semigroup $\{S(s) : s \geqslant 0\}$ in a reflexive Banach space with a uniformly Gâteaux differentiable norm, uniformly convex smooth space and uniformly convex Banach with a weakly continuous normalized duality mapping:

$$\begin{cases} w_0 \in \Upsilon, \\ w_{n+1} = m_n u + (1 - m_n) q_n, \\ q_n = m'_n w_n + (1 - m'_n) S(s_n) w_n, \end{cases} \quad \text{for all } n \geqslant 1, \tag{13}$$

where $u$ is an arbitrary (but fixed) element in $\Upsilon$, $\{m_n\}$, $\{m'_n\}$ and $\{s_n\}$ are sequences in $(0, 1)$ and $\Re^+$, respectively.

In [20]-[21], Zhang considered the following implicit iteration scheme and proved that it converges strongly the common fixed point of strictly pseudocontractive semigroups in reflexive Banach spaces:

$$\begin{cases} w_0 \in \Upsilon, \\ w_n = (1 - m_n) w_{n-1} + m_n S(s_n) w_n, \end{cases} \quad \text{for all } n \geqslant 1. \tag{14}$$

where $\{m_n\}$ is a sequence in [0,1] and $\{s_n\}$ is an increasing sequence in $[0, +\infty)$.

Many authors have studied (14) and proved its convergence to the common fixed points of nonexpansive semigroup, strictly pseudocontractive semigroup and pseudocontractive semigroup respectively (see for example Kim [4], Ofem et al. [9], Quan et al. [12], Thong [16]–[17] and the references there in).

Recently, Chang et al. [3] introduced the following Man-type iteration process:

$$\begin{cases} w_0 \in \Upsilon, \\ w_{n+1} = (1 - m_n) w_n + m_n S(s_n) w_n, \end{cases} \quad \text{for all } n \geqslant 1, \tag{15}$$

where $\{m_n\}$ is a sequence in [0,1] and $\{s_n\}$ is an increasing sequence in $[0, \infty)$. The authors proved that (15) converges strongly to the common fixed point of Lipschitzian and demicontraction semigroup $\{S(s) : s \geqslant 0\}$. Precisely, they proved the following theorem:

**Theorem 1.4** [3] Let $\Upsilon$ be a nonempty, closed and convex subset a real Banach space $B$. Let $\Im = \{S(s) : s \geqslant 0\} : \Upsilon \to \Upsilon$ be a Lipschitzian demicontraction semigroup with a bounded measurable function $L : [0, \infty) \to [0, \infty)$ and a bounded function $\lambda : [0, \infty) \to [0, \infty)$ such that

$$L := \sup_{s \geqslant 0} L(s) < \infty, \ \ \lambda := \inf_{s \geqslant 0} \lambda(s) > 0, \ \ F(\Im) := \bigcap_{s \geqslant 0} F(S(s)) \neq \emptyset.$$

Let $\{w_n\}$ be the sequence iteratively generated by (15), where where $\{m_n\}$ is a sequence in [0,1] and $\{s_n\}$ is an increasing sequence in $[0,\infty)$. Suppose there exists a compact subset $K$ of $B$ such that $\bigcup_{s\geqslant 0} S(s)(\Upsilon) \subset K$ and assume that the following conditions holds:

$(M_1)$ $\sum_{n=1}^{\infty} m_n = \infty$;

$(M_2)$ $\sum_{n=1}^{\infty} m_n^2 < \infty$;

$(M_2)$ for any bounded subset $D \in \Upsilon$,

$$\lim_{n\to\infty} \sup_{w\in D, t\in\Re^+} \|S(t+s_n)w - S(s_n)w\| = 0, \tag{16}$$

then $\{w_n\}$ converges strongly to a common fixed point in $F(\Im)$.

Theorem 1.4 extends and improves the corresponding results of Aleyner and Reich [1], Shioji and Takahshi [13], Suzuki [14] and Xu [19].

Motivated and inspired by the above results, we introduce the following implicit iteration process for the Lipschitzian $\alpha$-demicontractive semigroup $\Im = \{S(s) : s \geqslant 0\}$ in a real Banach space:

$$\begin{cases} w_0 \in \Upsilon, \\ w_n = (1-m_n-p_n)w_{n-1} + m_n S(s_n)q_n + p_n S(s_n)w_n, \\ q_n = (1-m'_n-p'_n-r'_n)w_{n-1} + m'_n w_n + p'_n S(s_n)w_{n-1} + r'_n S(s_n)w_n, \end{cases} \tag{17}$$

for all $n \geqslant 1$, where $\{m_n\}, \{p_n\}, \{m'_n\}, \{p'_n\}$ and $\{r'_n\}$ are real sequences in [0,1] such that $m_n + p_n \leqslant 1$ and $m'_n + p'_n + r'_n \leqslant 1$.

We have the following special cases of our new iterative scheme:

- If $p_n = m'_n = r'_n = 0$, then (17) reduces to

$$\begin{cases} w_0 \in \Upsilon, \\ w_n = (1-m_n)w_{n-1} + m_n S(s_n)q_n, \quad \text{for all}\, n \geqslant 1, \\ q_n = (1-p'_n)w_{n-1} + p'_n S(s_n)w_{n-1}, \end{cases} \tag{18}$$

  where $\{m_n\}$ and $\{p'_n\}$ are real sequences in [0,1].
- If $p_n = m'_n = p'_n = 0$, then (17) reduces to

$$\begin{cases} w_0 \in \Upsilon, \\ w_n = (1-m_n)w_{n-1} + m_n S(s_n)q_n, \quad \text{for all}\, n \geqslant 1, \\ q_n = (1-r'_n)w_{n-1} + r'_n S(s_n)w_n, \end{cases} \tag{19}$$

  where $\{m_n\}$ and $\{r'_n\}$ are real sequences in [0,1]. The scheme (19) is known as the modified implicit Ishikawa-type iteration process.
- If $m'_n = p'_n = r'_n = 0$, then (17) reduces to

$$\begin{cases} w_0 \in \Upsilon, \\ w_n = (1-m_n-p_n)w_{n-1} + m_n S(s_n)w_{n-1} + p_n S(s_n)w_n, \end{cases} \text{for all}\, n \geqslant 1,$$

  where $\{m_n\}$ and $\{p_n\}$ are real sequences in [0,1] such that $m_n + p_n \leqslant 1$.

- If $m_n = m'_n = p'_n = r'_n = 0$, then (17) reduces to

$$\begin{cases} w_0 \in \Upsilon, \\ w_n = (1-p_n)w_{n-1} + p_n S(s_n) w_n \end{cases} \text{for all } n \geqslant 1, \tag{20}$$

  where $\{p_n\}$ is a real sequence in [0,1]. Clearly, from (20) we can see that (14) is a special case of our new iterative scheme (17).
- If $p_n = m'_n = p'_n = r'_n = 0$, then (17) reduces to

$$\begin{cases} w_0 \in \Upsilon, \\ w_n = (1-m_n)w_{n-1} + m_n S(s_n) w_{n-1} \end{cases} \text{for all } n \geqslant 1, \tag{21}$$

  where $\{m_n\}$ is a real sequences in [0,1]. Obviously, from (21), it follows that (15) is a special case of our new iterative scheme (17)

The purpose of this paper is to prove strong convergence theorem of our new iteration process for the Lipschitzian $\alpha$-demicontractive semigroups in a real Banach space. Our results generalize, extend and improve several well known results in the existing literature. For instance, Theorem 1.4, which generalizes, extends and improves several recent results, is a special case of our theorem.

## 2. Preliminaries

The following lemmas will be useful in proving our main results.

**Lemma 2.1** Let $J : B \to 2^{B^*}$ be the normalized duality mapping. Then, for any $w, p \in B$, the following inequality holds:

$$\|w+p\| \leqslant \|w\|^2 + 2\langle p, j(w+p)\rangle, \text{ for all } j(w+p) \in J(w+p).$$

**Lemma 2.2** [18] Let $\{\Psi_n\}$ and $\{\Phi_n\}$ be two sequences of nonnegative real numbers satisfying the following inequality:

$$\Psi_n \leqslant (1+\Phi_n)\Psi_n, \ n \geqslant 1. \tag{22}$$

If $\sum_{n=1}^{\infty} \Phi_n < \infty$, then $\lim_{n\to\infty} \Psi_n$ exists. Additionally, if $\{\Psi_n\}$ has a subsequence $\{\Psi_{n_i}\}$ such that $\Psi_{n_i} \to 0$, then $\lim_{n\to\infty} \Psi_n = 0$.

## 3. Main Results

**Theorem 3.1** Let $\Upsilon$ be a nonempty closed convex subset of a real Banach space $B$. Let $\Im = \{S(s) : s \geqslant 0\} : \Upsilon \to \Upsilon$ be a Lipschitzian $\alpha$-demicontractive semigroup with a bounded measure function $L : [0,\infty) \to [0,\infty)$ and a bounded function $\lambda : [0,\infty) \to [0,\infty)$ such that

$$L := \sup_{s\geqslant 0} L(s) < \infty, \ \ \lambda := \inf_{s\geqslant 0} \lambda(s) > 0, \ \ F(\Im) := \bigcap_{s\geqslant 0} F(S(s)) \neq \emptyset.$$

Also, let $\{w_n\}$ be the sequence iteratively generated by (17), where $\{m_n\}$, $\{p_n\}$, $\{m'_n\}$, $\{p'_n\}$ and $\{r'_n\}$ are sequences in [0,1] such that $m_n + p_n \leqslant 1$ and $m'_n + p'_n + r'_n \leqslant 1$, and $\{s_n\}$ is an increasing sequence in $[0, \infty)$. Suppose there exists a compact subset $K$ of $B$ such that $\bigcup_{s \geqslant 0} S(s)(\Upsilon) \subset K$ and assume that the following conditions hold:

(i) $\sum_{n=0}^{\infty} (m_n + p_n) = \infty$;
(ii) $\sum_{n=0}^{\infty} (m_n + p_n)^2 < \infty$;
(iii) $\sum_{n=0}^{\infty} m_n m'_n < \infty$, $\sum_{n=0}^{\infty} m_n p'_n < \infty$, $\sum_{n=0}^{\infty} m_n r'_n < \infty$;
(iv) $m_n L(m'_n + r'_n L + p_n L) < 1$;
(v) for any bounded subset $K \in \Upsilon$,

$$\lim_{n\to\infty} \sup_{w \in K, s \in \Re^+} \|S(t + s_n)w - S(s_n)w\| = 0, \tag{23}$$

then $\{w_n\}$ converges strongly to a common fixed point in $F(\Im)$.

**Proof.** We divide the proof into five steps.
First, we show that the sequence $\{w_n\}$ defined by

$$\begin{aligned} w_n = (1 - m_n - p_n)w_{n-1} + m_n S(s_n)[(1 - m'_n - p'_n - r'_n)w_{n-1} + m'_n w_n \\ + p'_n S(s_n)w_{n-1} + r'_n S(s_n)w_n] + p_n S(s_n)w_n, \end{aligned}$$

for each $n \in \mathbb{N}$ and $w_0 \in \Upsilon$ is well defined. For each $n \in \mathbb{N}$, define a mapping $H_n : \Upsilon \to \Upsilon$ by

$$\begin{aligned} H_n(w) = (1 - m_n - p_n)w_{n-1} + m_n S(s_n)[(1 - m'_n - p'_n - r'_n)w_{n-1} + m'_n w \\ + p'_n S(s_n)w_{n-1} + r'_n S(s_n)w] + p_n S(s_n)w, \end{aligned}$$

$n \in \mathbb{N}, w \in \Upsilon$. Notice that

$$\begin{aligned} \|H_n(w) - H_n(q)\| &= m_n \|S(s_n)[(1 - m'_n - p'_n - r'_n)w_{n-1} + m'_n w + p'_n S(s_n)w_{n-1} + r'_n S(s_n)w] \\ &\quad + p_n S(s_n)w - S(s_n)[(1 - m'_n - p'_n - r'_n)w_{n-1} + m'_n q \\ &\quad + p'_n S(s_n)w_{n-1} + r'_n S(s_n)q] + p_n S(s_n)q\| \\ &\leqslant m_n L \|m'_n(w - q) + r'_n(S(s_n)w - S(s_n)q) + p_n(S(s_n)w - S(s_n)q)\| \\ &\leqslant m_n L(m'_n \|w - q\| + r'_n L\|w - q\| + p_n L\|w - q\|) \\ &= m_n L(m'_n + r'_n L + p_n L)\|w - q\|, \ \forall \ w, q \in \Upsilon. \end{aligned}$$

From the restriction (iv), we see that $H_n$ is a contraction mapping for each $n \in \mathbb{N}$. By Banach contraction principle, we see that there exists a unique point $w_n \in \Upsilon$ such that

$$\begin{aligned} w_n = (1 - m_n - p_n)w_{n-1} + m_n S(s_n)[(1 - m'_n - p'_n - r'_n)w_{n-1} + m'_n w_n \\ + p'_n S(s_n)w_{n-1} + r'_n S(s_n)w_n] + p_n S(s_n)w_n, \end{aligned}$$

$n \in \mathbb{N}, w_0 \in \Upsilon$. This implies that the sequence generated by the implicit iterative process

(17) is well defined and hence, it can be employed to approximate the common fixed point of $\alpha$-demicontraction semigroup.
Since the common fixed point set $F(\Im)$ is nonempty, let $\alpha z \in F(\Im)$. For each $\alpha z \in F(\Im)$, we now show that $\lim_{n\to\infty} \|w_n - \alpha z\|$ exists. Using (17) we have that

$$\begin{aligned}
\|q_n - \alpha z\| &= \|(1 - m'_n - p'_n - r'_n)w_{n-1} + m'_n w_n + p'_n S(s_n)w_{n-1} + r'_n S(s_n)w_n - \alpha z\| \\
&\leqslant (1 - m'_n - p'_n - r'_n)\|w_{n-1} - \alpha z\| + m'_n\|w_n - \alpha z\| \\
&\quad + p'_n\|S(s_n)w_{n-1} - \alpha z\| + r'_n\|S(s_n)w_n - \alpha z\| \\
&\leqslant \|w_{n-1} - \alpha z\| + m'_n\|w_n - \alpha z\| + p'_n L\|w_{n-1} - \alpha z\| + r'_n L\|w_n - \alpha z\| \\
&= (1 + p'_n L)\|w_{n-1} - \alpha z\| + (m'_n + r'_n L)\|w_n - \alpha z\| \\
&\leqslant (1 + L)\|w_{n-1} - \alpha z\| + (m'_n + r'_n L)\|w_n - \alpha z\|. \qquad (24)
\end{aligned}$$

Notice from (17) that

$$\begin{aligned}
\|q_n - w_n\| &= \|q_n - w_{n-1} + w_{n-1} - w_n\| \\
&\leqslant \|q_n - w_{n-1}\| + \|w_{n-1} - w_n\| \\
&= \|[(1 - m'_n - p'_n - r'_n)w_{n-1} + m'_n w_n + p'_n S(s_n)w_{n-1} + r'_n S(s_n)w_n] - w_{n-1}\| \\
&\quad + \|w_{n-1} - [(1 - m_n - p_n)w_{n-1} + m_n S(s_n)q_n + p_n S(s_n)w_n]\| \\
&= \|m'_n(w_n - w_{n-1}) + p'_n(S(s_n)w_{n-1} - w_{n-1}) + r'_n(S(s_n)w_n - w_{n-1})\| \\
&\quad + \|m_n(w_{n-1} - S(s_n)q_n) + p_n(w_{n-1} - S(s_n)w_n)\| \\
&\leqslant m'_n\|w_n - \alpha z\| + m'_n\|w_{n-1} - \alpha z\| + p'_n\|S(s_n)w_{n-1} - \alpha z\| + p'_n\|w_{n-1} - \alpha z\| \\
&\quad + r'_n\|S(s_n)w_n - \alpha z\| + r'_n\|w_{n-1} - \alpha z\| + m_n\|w_{n-1} - \alpha z\| \\
&\quad + m_n\|S(s_n)q_n - \alpha z\| + p_n\|w_{n-1} - \alpha z\| + p_n\|S(s_n)w_n - \alpha z\| \\
&\leqslant m'_n\|w_n - \alpha z\| + m'_n\|w_{n-1} - \alpha z\| + p'_n L\|w_{n-1} - \alpha z\| + p'_n\|w_{n-1} - \alpha z\| \\
&\quad + r'_n L\|w_n - \alpha z\| + r'_n\|w_{n-1} - \alpha z\| + m_n\|w_{n-1} - \alpha z\| + m_n L\|q_n - \alpha z\| \\
&\quad + p_n\|w_{n-1} - \alpha z\| + p_n L\|w_n - \alpha z\|. \qquad (25)
\end{aligned}$$

Putting (24) into (25) we obtain

$$\begin{aligned}
\|q_n - w_n\| &\leqslant m'_n\|w_n - \alpha z\| + m'_n\|w_{n-1} - \alpha z\| + p'_n L\|w_{n-1} - \alpha z\| + p'_n\|w_{n-1} - \alpha z\| \\
&\quad + r'_n L\|w_n - \alpha z\| + r'_n\|w_{n-1} - \alpha z\| + m_n\|w_{n-1} - \alpha z\| + m_n L[(1 - \alpha z\| \\
&\quad + L)\|w_{n-1} + (m'_n + r'_n L)\|w_n - \alpha z\|] + p_n\|w_{n-1} - \alpha z\| + p_n L\|w_n - \alpha z\| \\
&= [m'_n + p'_n + r'_n + p'_n L + m_n + m_n L(1 + L) + p_n]\|w_{n-1} - \alpha z\| \\
&\quad + [m'_n + r'_n L + m_n L(m'_n + r'_n L) + p_n L]\|w_n - \alpha z\| \\
&= [m'_n + p'_n(1 + L) + r'_n + m_n(1 + L(1 + L)) + p_n]\|w_{n-1} - \alpha z\| \\
&\quad + [m'_n(1 + m_n L) + r'_n L(1 + m_n L) + p_n L]\|w_n - \alpha z\| \\
&\leqslant [m'_n + p'_n(1 + L) + r'_n + m_n(1 + L(1 + L)) + p_n]\|w_{n-1} - \alpha z\| \\
&\quad + [m'_n(1 + L) + r'_n L(1 + L) + p_n L]\|w_n - \alpha z\|. \qquad (26)
\end{aligned}$$

Using (17) and Lemma 2.1, we have

$$\begin{aligned}
\|w_n - \alpha z\|^2 &= \|(1-m_n-p_n)w_{n-1} + m_n S(s_n)q_n + p_n S(s_n)w_n - \alpha z\|^2 \\
&= \|(1-m_n-p_n)(w_{n-1}-\alpha z) + m_n(S(s_n)q_n - \alpha z) + p_n(S(s_n)w_n - \alpha z)\|^2 \\
&\leqslant (1-m_n-p_n)^2\|w_{n-1}-\alpha z\|^2 + 2\langle m_n(S(s_n)q_n - \alpha z) \\
&\quad + p_n(S(s_n)w_n - \alpha z), j(w_n - \alpha z)\rangle \\
&= (1-m_n-p_n)^2\|w_{n-1}-\alpha z\|^2 + 2m_n\langle S(s_n)q_n - \alpha z, j(w_n-\alpha z)\rangle \\
&\quad + 2p_n\langle S(s_n)w_n - \alpha z, j(w_n - \alpha z)\rangle \\
&= (1-m_n-p_n)^2\|w_{n-1}-\alpha z\|^2 + 2m_n\langle S(s_n)q_n - S(s_n)w_n \\
&\quad + S(s_n)w_n - \alpha z, j(w_n-\alpha z)\rangle + 2p_n\langle S(s_n)w_n - \alpha z, j(w_n - \alpha z)\rangle \\
&= (1-m_n-p_n)^2\|w_{n-1}-\alpha z\|^2 + 2m_n\langle S(s_n)q_n - S(s_n)w_n, j(w_n-\alpha z)\rangle \\
&\quad + 2m_n\langle S(s_n)w_n - \alpha z, j(w_n-\alpha z)\rangle + 2p_n\langle S(s_n)w_n - \alpha z, j(w_n - \alpha z)\rangle \\
&\leqslant (1-m_n-p_n)^2\|w_{n-1}-\alpha z\|^2 + 2m_n\|S(s_n)q_n - S(s_n)w_n\|\|w_n-\alpha z\| \\
&\quad + 2(m_n+p_n)\langle S(s_n)w_n - \alpha z, j(w_n - \alpha z)\rangle \\
&\leqslant (1-m_n-p_n)^2\|w_{n-1}-\alpha z\|^2 + 2m_nL\|q_n - w_n\|\|w_n-\alpha z\| \\
&\quad + 2(m_n+p_n)\langle S(s_n)w_n - \alpha z, j(w_n - \alpha z)\rangle. \qquad (27)
\end{aligned}$$

Substituting (26) into (27), we obtain

$$\begin{aligned}
\|w_n - \alpha z\|^2 &\leqslant (1-m_n-p_n)^2\|w_{n-1}-\alpha z\|^2 + 2m_nL\{[m'_n + p'_n(1+L) + r'_n \\
&\quad + m_n(1+L(1+L)) + p_n]\|w_{n-1}-\alpha z\| + [m'_n(1+L) + r'_nL(1+L) \\
&\quad + p_nL]\|w_n - \alpha z\|\}\|w_n-\alpha z\| + 2(m_n+p_n)\langle S(s_n)w_n - \alpha z, j(w_n-\alpha z)\rangle \\
&= (1-m_n-p_n)^2\|w_{n-1}-\alpha z\|^2 + 2m_nL[m'_n + p'_n(1+L) + r'_n \qquad (28) \\
&\quad + m_n(1+L(1+L)) + p_n]\|w_{n-1}-\alpha z\|\|w_n-\alpha z\| + 2m_nL[m'_n(1+L) \\
&\quad + r'_nL(1+L) + p_nL]\|w_n-\alpha z\|^2 + 2(m_n+p_n)\langle S(s_n)w_n - \alpha z, j(w_n-\alpha z)\rangle.
\end{aligned}$$

Now, using the following well known inequality

$$\|w_{n-1}-\alpha z\|\|w_n-\alpha z\| \leqslant \frac{1}{2}(\|w_{n-1}-\alpha z\|^2 + \|w_n - \alpha z\|^2), \qquad (29)$$

then we obtain from (29) that

$$\begin{aligned}
\|w_n - \alpha z\|^2 &\leqslant (1-m_n-p_n)^2\|w_{n-1}-\alpha z\|^2 + 2m_nL[m'_n + p'_n(1+L) + r'_n + m_n(1 \\
&\quad + L(1+L)) + p_n] \times \frac{1}{2}(\|w_{n-1}-\alpha z\|^2 + \|w_n-\alpha z\|^2) + 2m_nL[m'_n(1+L) \\
&\quad + r'_nL(1+L) + p_nL]\|w_n-\alpha z\|^2 + 2(m_n+p_n)\langle S(s_n)w_n - \alpha z, j(w_n-\alpha z)\rangle
\end{aligned}$$

$$= \{(1-m_n-p_n)^2 + m_nL[m'_n + p'_n(1+L) + r'_n + m_n(1+L(1+L)) \\ +p_n]\}\|w_{n-1}-\alpha z\|^2 + \{m_nL[m'_n + p'_n(1+L) + r'_n + m_n(1+L(1+L)) + p_n] \\ +2m_nL[m'_n(1+L) + r'_nL(1+L) + p_nL]\}\|w_n-\alpha z\|^2 \\ +2(m_n+p_n)\langle S(s_n)w_n-\alpha z, j(w_n-\alpha z)\rangle \quad (30)$$

Since the semigroup $\Im = \{S(s) : s \geqslant 0\}$ is $\alpha$-demicontraction semigroup, then we obtain from (30) that

$$\|w_n-\alpha z\|^2 \leqslant \{(1-m_n-p_n)^2 + m_nL[m'_n + p'_n(1+L) \\ +r'_n + m_n(1+L(1+L)) + p_n]\}\|w_{n-1}-\alpha z\|^2 \\ +\{m_nL[m'_n + p'_n(1+L) + r'_n + m_n(1+L(1+L)) + p_n] \\ +2m_nL[m'_n(1+L) + r'_nL(1+L) + p_nL]\}\|w_n-\alpha z\|^2 \\ +2(m_n+p_n)\|w_n-\alpha z\|^2 - 2(m_n+p_n)\lambda\|w_n-S(s_n)w_n\|^2 \\ = \{(1-m_n-p_n)^2 + m_nL[m'_n + p'_n(1+L) \\ +r'_n + m_n(1+L(1+L)) + p_n]\}\|w_{n-1}-\alpha z\|^2 \\ +\{m_nL[m'_n + p'_n(1+L) + r'_n + m_n(1+L(1+L)) + p_n] \\ +2m_nL[m'_n(1+L) + r'_nL(1+L) + p_nL] + 2(m_n+p_n)\}\|w_n-\alpha z\|^2 \\ -2(m_n+p_n)\lambda\|w_n-S(s_n)w_n\|^2 \\ = \mu_n\|w_{n-1}-\alpha z\|^2 + \eta_n\|w_n-\alpha z\|^2 - 2(m_n+p_n)\lambda\|w_n-S(s_n)w_n\|^2, \quad (31)$$

where

$$\mu_n = (1-m_n-p_n)^2 + m_nL[m'_n + p'_n(1+L) + r'_n + m_n(1+L(1+L)) + p_n]. \\ \eta_n = m_nL[m'_n + p'_n(1+L) + r'_n + m_n(1+L(1+L)) + p_n] \\ +2m_nL[m'_n(1+L) + r'_nL(1+L) + p_nL] + 2(m_n+p_n).$$

By simplifying and transposing (31), we obtain

$$\|w_n-\alpha z\|^2 \leqslant \left[\frac{\mu_n}{1-\eta_n}\right]\|w_{n-1}-\alpha z\|^2 - \frac{2(m_n+p_n)\lambda}{1-\eta_n}\|w_n-S(s_n)w_n\|^2 \\ = \left[1+\frac{\delta_n}{1-\eta_n}\right]\|w_{n-1}-\alpha z\|^2 \\ -\frac{2(m_n+p_n)\lambda}{1-\eta_n}\|w_n-S(s_n)w_n\|^2, \quad (32)$$

where

$$\delta_n = \mu_n + \eta_n - 1 = (m_n+p_n)^2 + 2m_nL[m'_n + p'_n(1+L) + r'_n \\ +m_n(1+L(1+L)) + p_n] + 2m_nL[m'_n(1+L) + r'_nL(1+L) + p_nL].$$

Since from condition (ii) we have $\sum_{n=0}^{\infty}(m_n+p_n)^2 < \infty$, this implies that $m_n+p_n, m_n^2$ and

$m_np_n \to 0$ as $n \to \infty$ and also, it follows from condition (iii) that

$$\eta_n = m_nL[m'_n + p'_n(1+L) + r'_n + m_n(1 + L(1+L)) + p_n] \\ +2m_nL[m'_n(1+L) + r'_nL(1+L) + p_nL] + 2(m_n + p_n) \to 0 \text{ as } n \to \infty,$$

therefore, there exists a positive integer $n_0$ such that

$$1 - \eta_n \geqslant \frac{1}{2}, \text{ for any } \quad n \geqslant n_0. \tag{33}$$

From (32), we obtain

$$\begin{aligned} \|w_n - \alpha z\|^2 &\leqslant [1 + 2\delta_n]\,\|w_{n-1} - \alpha z\|^2 - 2(m_n + p_n)\lambda\|w_n - S(s_n)w_n\|^2 \\ &= [1 + \varPhi_n]\,\|w_{n-1} - \alpha z\|^2 - 2(m_n + p_n)\lambda\|w_n - S(s_n)w_n\|^2 \qquad (34) \\ &\leqslant [1 + \varPhi_n]\,\|w_{n-1} - \alpha z\|^2, \qquad (35) \end{aligned}$$

where $\varPhi_n = 2\delta_n$. Since from condition (ii) we have $\sum_{n=0}^{\infty}(m_n + p_n)^2 < \infty$, this implies that $\sum_{n=0}^{\infty} m_n^2 < \infty$ and $\sum_{n=0}^{\infty} m_np_n < \infty$, with the help of condition (iii) we have that $\sum_{n=0}^{\infty} \varPhi_n < \infty$. It follows from (35) and Lemma 2.2 that $\lim_{n\to\infty} \|w_n - \alpha z\|^2$ exists and also $\lim_{n\to\infty} \|w_n - \alpha z\|$ exists, therefore $\{w_n\}$ is bounded in $C$.
Now we prove that $\liminf_{n\to\infty} \|w_n - S(s_n)w_n\| = 0$. Notice from (34) that

$$2(m_n + p_n)\lambda\|w_n - S(s_n)w_n\|^2 \leqslant \|w_{n-1} - \alpha z\|^2 - \|w_n - \alpha z\|^2 + K^2\varPhi_n, \tag{36}$$

for all $n \geqslant n_0$, where $K = \sup_{n\geqslant 0} \|w_{n-1} - \alpha z\|$. Thus, from (36) we obtain

$$2\lambda \sum_{j=n_0+1}^{\infty} (m_j + p_j)\|w_j - S(s_j)w_j\|^2 \leqslant \|w_{n_0} - \alpha z\|^2 + K^2 \sum_{j=n_0+1}^{\infty} \varPhi_j, \tag{37}$$

and hence,

$$2\lambda \sum_{n=1}^{\infty}(m_n + p_n)\|w_n - S(s_n)w_n\|^2 \leqslant \|w_{n_0} - \alpha z\|^2 + K^2 \sum_{n=1}^{\infty} \varPhi_n. \tag{38}$$

Since $\sum_{n=1}^{\infty} \varPhi_n < \infty$, then it follows from (38) that

$$\sum_{n=1}^{\infty}(m_n + p_n)\|w_n - S(s_n)w_n\|^2 < \infty. \tag{39}$$

Since $\sum_{n=1}^{\infty}(m_n + p_n) = \infty$, then we obtain from (39) that

$$\liminf_{n\to\infty} \|w_n - S(s_n)w_n\| = 0. \tag{40}$$

Finally, we show that the sequence $\{w_n\}$ iteratively generated by (17) converges strongly to a common fixed point in $F(\Im)$.

From (40), we have that $\liminf_{n\to\infty} \|w_n - S(s_n)w_n\| = 0$. Also, by assumption, this follows that there exists a compact subset $K$ of $B$ such that $\bigcup_{s\geqslant 0} S(s)(\Upsilon) \subset K$ and so subsequence $\{w_{n_i}\}$ of $\{w_n\}$ exists such that

$$\lim_{n_k\to\infty} \|w_{n_k} - S(s_{n_k})w_{n_k}\| = 0, \ \lim_{n_k\to\infty} S(s_{n_k})w_{n_k} = \alpha g, \tag{41}$$

for some $\alpha g \in \Upsilon$. Hence, from (41) we have that $w_{n_k} \to \alpha g$ as $n_k \to \infty$.

We now show that

$$\lim_{n_k\to\infty} \|S(s)w_{n_k} - w_{n_k}\| = 0, \text{ for all } s \geqslant 0. \tag{42}$$

Notice from the condition (v) and (41) that for any $s > 0$,

$$\begin{aligned}
\|S(s)w_{n_k} - w_{n_k}\| &\leqslant \|S(s)w_{n_k} - S(s+s_{n_k})w_{n_k}\| + \|S(s+s_{n_k})w_{n_k} - S(s_{n_k})w_{n_k}\| \\
&\quad + \|S(s_{n_k})w_{n_k} - w_{n_k}\| \\
&\leqslant (1+L)\|w_{n_k} - S(s_{n_k})w_{n_k}\| + \|S(s+s_{n_k})w_{n_k} - S(s_{n_k})w_{n_k}\| \\
&\leqslant (1+L)\|w_{n_k} - S(s_{n_k})w_{n_k}\| \\
&\quad + \sup_{\upsilon\in\{w_n\}, t\in\Re^+} \|S(t+s_n)\upsilon - S(s_n)\upsilon\| \to 0
\end{aligned} \tag{43}$$

as $n_k \to \infty$. Since $w_{n_k} \to \alpha g$ as $n_k \to \infty$ and the semigroup $\Im = \{S(s) : s \geqslant 0\}$ is Lipschitzian, then we have from (43) that $\alpha g = S(s)(\alpha g)$ for all $s \geqslant 0$, that is,

$$\alpha g \in F(\Im) = \bigcap_{s\geqslant 0} F(S(s)). \tag{44}$$

Knowing that $w_{n_k} \to \alpha g$ as $n_k \to \infty$ and the limit $\lim_{n\to\infty} \|w_n - \alpha z\|$ exists, this implies that $w_n \to \alpha g \in F(\Im)$ as $n \to \infty$. This completes the proof of Theorem 3.1. ■

The following results are obtained immediately from Theorem 3.1.

**Corollary 3.2** Let $\Upsilon$ be a nonempty closed convex subset of a real Banach space $B$. Let $\Im = \{S(s) : s \geqslant 0\} : \Upsilon \to \Upsilon$ be a Lipschitzian $\alpha$-demicontractive semigroup with a bounded measure function $L : [0,\infty) \to [0,\infty)$ and a bounded function $\lambda : [0,\infty) \to [0,\infty)$ such that

$$L := \sup_{s\geqslant 0} L(s) < \infty, \ \ \lambda := \inf_{s\geqslant 0} \lambda(s) > 0, \ \ F(\Im) := \bigcap_{s\geqslant 0} F(S(s)) \neq \emptyset.$$

Also, let $\{w_n\}$ be the sequence iteratively generated by (18), where $\{m_n\}$ and $\{p'_n\}$ are sequences in [0,1], and $\{s_n\}$ is an increasing sequence in $[0, +\infty)$. Suppose there exists a compact subset $K$ of $B$ such that $\bigcup_{s\geqslant 0} S(s)(\Upsilon) \subset K$ and assume that the following conditions hold:

(i) $\sum_{n=0}^{\infty} m_n = \infty$;
(ii) $\sum_{n=0}^{\infty} m_n^2 < \infty$;
(iii) $\sum_{n=0}^{\infty} m_n p'_n < \infty$;
(iv) for any bounded subset $K \in \Upsilon$,

$$\lim_{n\to\infty} \sup_{w\in K, s\in\Re^+} \|S(t+s_n)w - S(s_n)w\| = 0, \tag{45}$$

then $\{w_n\}$ converges strongly to a common fixed point in $F(\Im)$.

**Proof.** Set $p_n = m'_n = r'_n = 0$ in Theorem 3.1 ■

**Corollary 3.3** Let $\Upsilon$ be a nonempty closed convex subset of a real Banach space $B$. Let $\Im = \{S(s) : s \geqslant 0\} : \Upsilon \to \Upsilon$ be a Lipschitzian $\alpha$-demicontractive semigroup with a bounded measure function $L : [0, \infty) \to [0, \infty)$ and a bounded function $\lambda : [0, \infty) \to [0, \infty)$ such that

$$L := \sup_{s\geqslant 0} L(s) < \infty, \ \ \lambda := \inf_{s\geqslant 0} \lambda(s) > 0, \ \ F(\Im) := \bigcap_{s\geqslant 0} F(S(s)) \neq \emptyset.$$

Also, let $\{w_n\}$ be the sequence iteratively generated by (19), where $\{m_n\}$ and $\{r'_n\}$ are sequences in [0,1] and $\{s_n\}$ is an increasing sequence in $[0, \infty)$. Suppose there exists a compact subset $K$ of $B$ such that $\bigcup_{s\geqslant 0} S(s)(\Upsilon) \subset K$ and assume that the following conditions hold:

(i) $\sum_{n=0}^{\infty} m_n = \infty$;
(ii) $\sum_{n=0}^{\infty} m_n^2 < \infty$;
(iii) $\sum_{n=0}^{\infty} m_n r'_n < \infty$;
(iv) $m_n r'_n L^2 < 1$;
(v) for any bounded subset $K \in \Upsilon$,

$$\lim_{n\to\infty} \sup_{w\in K, s\in\Re^+} \|S(t+s_n)w - S(s_n)w\| = 0, \tag{46}$$

then $\{w_n\}$ converges strongly to a common fixed point in $F(\Im)$.

**Proof.** Set $p_n = m'_n = p'_n = 0$ in Theorem 3.1. ■

**Corollary 3.4** Let $\Upsilon$ be a nonempty closed convex subset of a real Banach space $B$. Let $\Im = \{S(s) : s \geqslant 0\} : \Upsilon \to \Upsilon$ be a Lipschitzian $\alpha$-demicontractive semigroup with a bounded measure function $L : [0, \infty) \to [0, \infty)$ and a bounded function $\lambda : [0, \infty) \to$

$[0, \infty)$ such that

$$L := \sup_{s \geqslant 0} L(s) < \infty, \;\; \lambda := \inf_{s \geqslant 0} \lambda(s) > 0, \;\; F(\Im) := \bigcap_{s \geqslant 0} F(S(s)) \neq \emptyset.$$

Also, let $\{w_n\}$ be the sequence iteratively generated by (20), where $\{m_n\}$ and $\{p_n\}$ are sequences in [0,1] such that $m_n + p_n \leqslant 1$ and $\{s_n\}$ is an increasing sequence in $[0, \infty)$. Suppose there exists a compact subset $K$ of $B$ such that $\bigcup_{s \geqslant 0} S(s)(\Upsilon) \subset K$ and assume that the following conditions hold:

(i) $\sum_{n=0}^{\infty} (m_n + p_n) = \infty$;
(ii) $\sum_{n=0}^{\infty} (m_n + p_n)^2 < \infty$;
(iii) $m_n p_n L^2 < 1$;
(iv) for any bounded subset $K \in \Upsilon$,

$$\lim_{n \to \infty} \sup_{w \in K, s \in \Re^+} \|S(t + s_n)w - S(s_n)w\| = 0, \tag{47}$$

then $\{w_n\}$ converges strongly to a common fixed point in $F(\Im)$.

**Proof.** Set $m'_n = p'_n = r'_n$ in Theorem 3.1. ■

**Corollary 3.5** Let $\Upsilon$ be a nonempty closed convex subset of a real Banach space $B$. Let $\Im = \{S(s) : s \geqslant 0\} : \Upsilon \to \Upsilon$ be a Lipschitzian $\alpha$-demicontractive semigroup with a bounded measure function $L : [0, \infty) \to [0, \infty)$ and a bounded function $\lambda : [0, \infty) \to [0, \infty)$ such that

$$L := \sup_{s \geqslant 0} L(s) < \infty, \;\; \lambda := \inf_{s \geqslant 0} \lambda(s) > 0, \;\; F(\Im) := \bigcap_{s \geqslant 0} F(S(s)) \neq \emptyset.$$

Also, let $\{w_n\}$ be the sequence iteratively generated by (20), where $\{p_n\}$ is a sequence in [0,1] and $\{s_n\}$ is an increasing sequence in $[0, \infty)$. Suppose there exists a compact subset $K$ of $B$ such that $\bigcup_{s \geqslant 0} S(s)(\Upsilon) \subset K$ and assume that the following conditions hold:

(i) $\sum_{n=0}^{\infty} p_n = \infty$;
(ii) $\sum_{n=0}^{\infty} p_n^2 < \infty$;
(iii) for any bounded subset $K \in \Upsilon$,

$$\lim_{n \to \infty} \sup_{w \in K, s \in \Re^+} \|S(t + s_n)w - S(s_n)w\| = 0, \tag{48}$$

then $\{w_n\}$ converges strongly to a common fixed point in $F(\Im)$.

**Proof.** Set $m_n = m'_n = p'_n = r'_n = 0$ in Corollary 3.4. ■

**Corollary 3.6** Let $\Upsilon$ be a nonempty closed convex subset of a real Banach space $B$. Let $\Im = \{S(s) : s \geqslant 0\} : \Upsilon \to \Upsilon$ be a Lipschitzian $\alpha$-demicontractive semigroup with a bounded measure function $L : [0,\infty) \to [0,\infty)$ and a bounded function $\lambda : [0,\infty) \to [0,\infty)$ such that

$$L := \sup_{s\geqslant 0} L(s) < \infty, \ \ \lambda := \inf_{s\geqslant 0} \lambda(s) > 0, \ \ F(\Im) := \bigcap_{s\geqslant 0} F(S(s)) \neq \emptyset.$$

Also, let $\{w_n\}$ be the sequence iteratively generated by (20), where $\{m_n\}$ is a sequence in [0,1] and $\{s_n\}$ is an increasing sequence in $[0,\infty)$. Suppose there exists a compact subset $K$ of $B$ such that $\bigcup_{s\geqslant 0} S(s)(\Upsilon) \subset K$ and assume that the following conditions hold:

(i) $\sum_{n=0}^{\infty} m_n = \infty$;
(ii) $\sum_{n=0}^{\infty} m_n^2 < \infty$;
(iii) for any bounded subset $K \in \Upsilon$,

$$\lim_{n\to\infty} \sup_{w\in K, s\in\Re^+} \|S(t+s_n)w - S(s_n)w\| = 0, \tag{49}$$

then $\{w_n\}$ converges strongly to a common fixed point in $F(\Im)$.

**Proof.** Set $p_n = m'_n = p'_n = r'_n = 0$ in Corollary 3.4. ■

***Remark 3*** *If we set $\alpha = 1$ in Corollary 3.6, then we capture completely the result of Chang et al. [3].*

This is just but a few of the numerous results that can be obtain from Theorem 3.1.

## 4. Conclusion

Since the class of $\alpha$-demicontractive semigroup is more general than of all nonexpasive semigroup, strictly pseudocontractive semigroup, demicontractive semigroup and also, owing to the fact that our new iterative scheme properly includes those considered in [2, 3, 13, 14, 19, 20, 22]. Hence our results extend, generalize, improve and unify the corresponding results in [2, 3, 13, 14, 19–22] and also, give affirmative answers the open questions raised by Suzuki [14] and Xu [19].


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## References


[1] A. Aleyner, S. Reich, An explicit construction of sunny nonexpansive retractions in Banach spaces, Fixed Point Theory Appl. 3 (2005), 295-305.
[2] S. S. Chang, C. K. Chan, H. W. Joseph Lee, L. Yang, A system of mixed equilibrium problems, fixed point problems of strictly pseudocontractive mappings and nonexpansive semigroups, Appl. Math. Comput. 216 (1) (2010), 51-60.
[3] S. S. Chang, Y. J. Cho, H. W. Joseph Lee, C. K. Chan, Strong convergence theorems for Lipschitzian demicontraction semigroups in Banach spaces, Fixed Point Theory Appl. 2011, 2011:583423.

[4] G. E. Kim, Approximating common fixed points of Lipschitzian pseudocontraction semigroups, J. Fixed Point Theory Appl. 18 (2016), 927-934.
[5] S. Li, L. H. Li, F. Su, General iteration methods for a one-parameter nonexpansive semigroups in Hilbert spaces, Nonlinear Anal. 70 (9) (2009), 3065-3071.
[6] L. Măruşter, S. Măruşter, Strong convergence of the Mann iteration for $\alpha$-demicontractive mappings, Math. Comput. Modelling. 54 (2011), 2486-2492.
[7] A. E. Ofem, Strong convergence of a multi-step implicit iterative scheme with errors for common fixed points of uniformly L–Lipschitzian total asymptotically strict pseudocontractive mappings, Results in Nonlinear Anal. 3 (2) (2020), 100-116.
[8] A. E. Ofem, Strong convergence of modified implicit hybrid S-iteration scheme for finite family of nonexpansive and asymptotically generalized $\Phi$-hemicontractive mappings, Malaya J. Matematik. 8 (4) (2020), 1643-1649.
[9] A. E. Ofem, D. I. Igbokwe, X. A. Udo-utun, Implicit iteration process for Lipschitzian $\alpha$-hemicontraction semigroups, MathLAB J. 7 (2020), 43-52.
[10] A. E. Ofem, U. E. Udofia, Iterative solutions for common fixed points of nonexpansive mappings and strogly mappings and strongly pesudocontractive mappings with applications, Canad. J. Appl. Math. 3 (1) (2021), 18-36.
[11] M. O. Osilike, A. C. Onah, Strong convergence of the Ishikawa Iteration for Lipschitz Hemicontractive mappings, Annals of West University of Timisoara (Seria Mathematica Informatica). LIII 1 (2015), 151-161.
[12] J. Quan, S. Chang, M. Liu, Strong and weak convergence of an implicit iterative process for pseudocontractive semigroups in Banach space, Fixed Point Theory Appl. 2012, 2012:16.
[13] N. Shioji, W. Takahashi, Strong convergence theorems for asymptotically nonexpansive mappings in Hilbert spaces, Nonlinear Anal. 34 (1) (1998), 87-99.
[14] T. Suzuki, On strong convergence to a common fixed point of nonexpansive semigroups in Hilbert spaces, Proc. Am. Math. Soc. 131 (7) (2003), 2133-2136.
[15] T. Suzuki, Fixed point property for nonexpansive mappings versus that for nonexpansive semigroups, Nonlinear Anal. 70 (2009), 3358-3361.
[16] D. V. Thong, An implicit iteration process for nonexpansive semigroups, Nonlinear Anal. 74 (2011), 6116-6120.
[17] D. V. Thong, On Mann type implicit iteration process for strictly pseudocontraction semigroups, Annals of the University of Craiova, Math. Comput Sci. Ser. 38 (3) (2011), 101-108.
[18] H. K. Xu, Inequalities in Banach spaces with applications, Nonlinear Anal. 16 (12) (1991), 1127-1138.
[19] H. K. Xu, Strong convergence theorem for contraction semigroups in Banach spaces, Bull. Aust. Math. Soc. 72 (3) (2005), 371-379.
[20] S. S. Zhang, Convergence theorem of common fixed points for Lipschitzian pseudo-contraction semi-groups in Banach spaces, Appl. Math. Mech. 30 (2009), 145-152.
[21] S. S. Zhang, Weak convergence theorem for Lipschitzian pseudocontraction semigroups in Banach spaces, Acta. Math. Sinica. 26 (2010), 337-344.
[22] S. S. Zhang, L. Yang, J. A. Liu, Strong convergence theorem for nonexpansive semigroups in Banach spaces, Appl. Math. Mech. 28 (10) (2007), 1287-1297.