

## Conjectures on the anti-automorphism of $Z$ -basis of the Steenrod algebra

T. Vergili <sup>a,\*</sup>, I. Karaca <sup>b</sup>

<sup>a</sup>*Department of Mathematics, Faculty of Science, Karadeniz Technical University, Trabzon, Turkey.*

<sup>b</sup>*Department of Mathematics, Faculty of Science, Ege University, Izmir, Turkey.*

Received 13 May 2020; Revised 9 March 2021; Accepted 11 March 2021.

Communicated by Hamidreza Rahimi

---

**Abstract.** In this paper, we compute the images of some of the  $Z$ -basis elements under the anti-automorphism map  $\chi$  of the mod 2 Steenrod algebra  $\mathcal{A}_2$  and propose some conjectures based on our computations.

© 2021 IAUCTB. All rights reserved.

---

**Keywords:** Steenrod Algebra, cohomology operations, conjugation map, Hopf Algebra.

**2010 AMS Subject Classification:** 55S10, 57T05.

### 1. Introduction

The mod 2 Steenrod algebra,  $\mathcal{A}_2$ , is the algebra generated by all stable primary cohomology operations, called the Steenrod operations, which play a crucial role in the solution of many problems, such as calculating homotopy groups of  $n$ -sphere, Hopf invariant problem, and characteristic classes of vector bundles in algebraic topology. Its Hopf algebraic structure allows us to define an anti-automorphism on it. Milnor constructed a base system on the Steenrod algebra and gave a formula for the images of the Milnor basis elements under the anti-automorphism map [7]. Through this formula, Davis [4] and Silverman [9] compute the images of some certain Steenrod operations, Barrat and Miller [3] obtain new identities related to the anti-automorphism. For prime  $p > 2$ , the mod  $p$  Steenrod algebra has also been studied in the literature [10] and we refer to [5, 6, 11] for the computations of the images of some certain Steenrod operations under the anti-automorphism map.

---

E-mail address: tane.vergili@ktu.edu.tr (T. Vergili); ismet.karaca@ege.edu.tr (I. Karaca).

There is no general rule for the computation of the elements in  $\mathcal{A}_2$  under the anti-automorphism, that means each computation is important on its own. In this paper, we consider some certain finite subalgebras of the mod 2 Steenrod algebra and compute the images of some of their basis elements under the anti-automorphism map and propose conjectures based on our computations.

## 2. Preliminaries

The Steenrod squares,  $Sq^i$  for  $i \geq 0$ , are group homomorphisms

$$Sq^i : H^n(X; \mathbb{Z}_2) \rightarrow H^{n+i}(X; \mathbb{Z}_2)$$

on the cohomology of the topological space  $X$ , where  $Sq^0$  is the identity and  $Sq^i$  is zero whenever  $n < i$  [10]. These squares satisfy the Adem relations

$$Sq^i Sq^j = \sum_{k=0}^{\lfloor \frac{i}{2} \rfloor} \binom{j-k-1}{i-2k} Sq^{i+j-k} Sq^k \quad (1)$$

for  $0 < i < 2j$ , where  $\lfloor \frac{i}{2} \rfloor$  denotes the greatest integer less than or equal to  $\frac{i}{2}$  and the binomial coefficients are taken modulo 2. The grading of the  $Sq^i$  is  $i$ , and for the composition of the Steenrod squares  $Sq^{i_1} Sq^{i_2} \dots Sq^{i_n}$ , the grading is  $i_1 + i_2 + \dots + i_n$ . The length of the monomial  $Sq^{i_1} Sq^{i_2} \dots Sq^{i_n}$  is the number of Steenrod squares,  $n$ . Note that the Adem relations preserve the grading and do not extend the length of the composition when it is applied on  $Sq^i Sq^j$  for  $0 < i < 2j$ .

These operations form a free graded associative  $\mathbb{Z}_2$ -algebra, called the mod 2 Steenrod algebra  $\mathcal{A}_2$  subject to the Adem relations. Note that  $\mathcal{A}_2$  is generated algebraically by  $Sq^{2^k}$  for every non-negative integer  $k$  [8]. For details of the Steenrod algebra, we refer to [1, 10, 12, 13].

$\mathcal{A}_2$  has a Hopf algebraic structure so it admits a unique anti-automorphism  $\chi$  on itself  $\chi : \mathcal{A}_2 \rightarrow \mathcal{A}_2$  defined on the Steenrod squares by

$$\chi(Sq^k) = \sum_{1 \leq i \leq k} Sq^i \chi(Sq^{k-i})$$

such that  $\chi(Sq^0) = Sq^0$  and  $\chi^2 = 1$  [7]. We will pay attention to the finite subalgebras  $\mathcal{A}_2(n)$  of  $\mathcal{A}_2$  which are algebraically generated by  $Sq^{2^i}$  for  $0 \leq i \leq n$  [7]. Note that each subalgebra  $\mathcal{A}_2(n)$  is a vector subspace of  $\mathcal{A}_2$  over  $\mathbb{Z}_2$  with dimension  $2^{(n+1)(n+2)/2}$ . Wood [14] defines  $Z$ -basis on  $\mathcal{A}_2$  that can be also extended to the whole algebra  $\mathcal{A}_2$ . Consider the monomial  $X_n = Sq^{1.2^n} Sq^{3.2^{n-1}} Sq^{7.2^{n-2}} \dots Sq^{2^{n+1}-1}$  for an integer  $n \geq 0$  and define  $Z_n = X_n X_{n-1} \dots X_1 X_0$ .

For instance,  $Z_0 = Sq^1$ ,  $Z_1 = Sq^2 Sq^3 Sq^1$  and  $Z_2 = Sq^4 Sq^6 Sq^7 Sq^2 Sq^3 Sq^1$ .

Wood [14] shows that  $Z_n$  is the top element, the monomial with a maximum length, of  $\mathcal{A}_2(n)$  and the set of  $2^{(n+1)(n+2)/2}$  monomials obtained by selecting all subsets of  $Z_n$  in the given order together with  $Sq^0$  is an additive basis for  $\mathcal{A}_2(n)$ . For instance, the basis elements for  $\mathcal{A}_2(2)$  are  $Sq^2 Sq^3 Sq^1$ ,  $Sq^2 Sq^3$ ,  $Sq^2 Sq^1$ ,  $Sq^3 Sq^1$ ,  $Sq^2$ ,  $Sq^3$ ,  $Sq^1$ ,  $Sq^0$ .

Now we compute the images of some of the basis elements under the anti-automorphism map  $\chi$ . So, we will use the stripping technique which is practical to derive a new relation

from old relation considering the Adem relations. It is based on the action of the dual Steenrod algebra to the original algebra  $\mathcal{A}_2$ . We refer to [15] for the details.

The basic idea of the stripping technique is as follows [15]. We call a finite non-negative sequence  $(\omega_1, \omega_2, \dots, \omega_k)$  an allowable vector if all  $\omega_i$ 's are some power of 2 and  $\omega_i \geq \omega_{i+1}$  for all  $i = 1, 2, \dots, k - 1$ . For example, the vectors  $(16, 4, 2)$ ,  $(4, 1)$ ,  $(1)$ , and  $(8, 4, 2, 1)$  are allowable while  $(16, 0, 4)$  and  $(2, 8)$  are not allowable. To obtain a new relation from old relation, we use these types of vectors and apply them to each of the monomial in the relation. Let  $Sq^{n_1}Sq^{n_2} \dots Sq^{n_m}$  be a monomial in the relation and  $(\omega_1, \omega_2, \dots, \omega_k)$  be the allowable vector. Stripping the relation with the vector is to strip each monomial in the relation and this is done by subtracting  $(\omega_1, \omega_2, \dots, \omega_k)$  componentwise from the vector of the gradings  $(n_1, n_2, \dots, n_m)$  of the monomial. For the case  $n \neq k$ , we will add 0's to make the length of the sequence equal, and for the case  $n_s - \omega_s < 0$ , we do not subtract. Then the new monomial derived from the original will be a monomial in the new relation. For instance, if the relation

$$Sq^{15}Sq^8 = 0 \tag{2}$$

is stripped by the allowable vector  $(4, 2)$ , then we get the new relation  $Sq^{11}Sq^6 = 0$ . To strip the relation given in (2) with the allowable vector  $(4)$ , first we add 0 to the sequence  $(4)$  to make the length equal so we have two possible vectors  $(4, 0)$  and  $(0, 4)$  and then we strip the relation with both to get  $Sq^{11}Sq^8 + Sq^{15}Sq^4 = 0$ .

The following theorem obtained from the stripping technique gives us a new relation in  $\mathcal{A}_2$ .

**Theorem 2.1** For  $n > 1$ ,  $Sq^{2^n-1}Sq^{2^n}Sq^{2^{n+1}} = Sq^{2^{n+2}-1}$ .

**Proof.** By the Adem relations, we have

$$Sq^{2^k-1}Sq^k = 0 \tag{3}$$

for  $k > 1$  [15]. If we take  $k = 2^n$ , then we get

$$Sq^{2^{n+1}-1}Sq^{2^n} = 0. \tag{4}$$

If we strip the relation (4) with the allowable vector  $(2^n)$ , we get

$$Sq^{2^n-1}Sq^{2^n} + Sq^{2^{n+1}-1} = 0. \tag{5}$$

Next, we take  $k = 2^{n+1}$  in the relation (3), then the relation transforms into

$$Sq^{2^{n+2}-1}Sq^{2^{n+1}} = 0.$$

Then we strip (6) with the allowable vector  $(2^{n+1})$  to get

$$Sq^{2^{n+1}-1}Sq^{2^{n+1}} + Sq^{2^{n+2}-1} = 0. \tag{6}$$

The above relation together with the relation (5) gives the desired relation

$$Sq^{2^n-1}Sq^{2^n}Sq^{2^{n+1}} = Sq^{2^{n+2}-1}.$$

■

### 3. Computations

The anti-automorphism  $\chi$  is compatible with the subalgebras  $\mathcal{A}_2(n)$  for  $n \geq 0$ . Since  $\chi$  does not change the degree of the monomial, we will derive conjectures for the images of the top element  $Z_n$  in  $\mathcal{A}_2(n)$  under  $\chi$ . Since  $Z_n = X_n Z_{n-1}$ , we start with computing  $\chi(X_n)$  for  $n = 0, 1, 2, 3$ . The computations in the following examples are obtained from the definition of  $\chi$  and the Adem relations.

**Example 3.1** It is trivial that  $\chi(X_0) = \text{Sq}^1$  and

$$\chi(X_1) = \chi(\text{Sq}^2 \text{Sq}^3) = \chi(\text{Sq}^3) \chi(\text{Sq}^2) = (\text{Sq}^2 \text{Sq}^1) \text{Sq}^2 = \text{Sq}^5 = \text{Sq}^2 \text{Sq}^3.$$

**Example 3.2** For the monomial  $X_2 = \text{Sq}^4 \text{Sq}^6 \text{Sq}^7$ , the image of  $X_2$  under  $\chi$  can be obtained as follows:

$$\begin{aligned} \chi(X_2) &= \chi(\text{Sq}^4 \text{Sq}^6 \text{Sq}^7) = \chi(\text{Sq}^7) \chi(\text{Sq}^6) \chi(\text{Sq}^4) \\ &= (\text{Sq}^4 \text{Sq}^2 \text{Sq}^1) (\text{Sq}^4 \text{Sq}^2) (\text{Sq}^3 \text{Sq}^1 + \text{Sq}^4) \\ &= \text{Sq}^4 \text{Sq}^2 \text{Sq}^1 \text{Sq}^4 \text{Sq}^2 \text{Sq}^3 \text{Sq}^1 + \text{Sq}^4 \text{Sq}^2 \text{Sq}^1 \text{Sq}^4 \text{Sq}^2 \text{Sq}^4 \\ &= 0 + \text{Sq}^{12} \text{Sq}^4 \text{Sq}^1 + \text{Sq}^{14} \text{Sq}^2 \text{Sq}^1 + \text{Sq}^{15} \text{Sq}^2 + \text{Sq}^{16} \text{Sq}^1 + \text{Sq}^{17} + \text{Sq}^{12} \text{Sq}^5 \\ &= \text{Sq}^4 \text{Sq}^6 \text{Sq}^7. \end{aligned}$$

**Example 3.3** Similar to Example 3.2, the image of the monomial  $X_3 = \text{Sq}^8 \text{Sq}^{12} \text{Sq}^{14} \text{Sq}^{15}$  under  $\chi$  is obtained as follows:

$$\begin{aligned} \chi(X_3) &= \chi(\text{Sq}^8 \text{Sq}^{12} \text{Sq}^{14} \text{Sq}^{15}) = \chi(\text{Sq}^{15}) \chi(\text{Sq}^{14}) \chi(\text{Sq}^{12}) \chi(\text{Sq}^8) \\ &= (\text{Sq}^8 \text{Sq}^4 \text{Sq}^2 \text{Sq}^1) (\text{Sq}^8 \text{Sq}^4 \text{Sq}^2) (\text{Sq}^8 \text{Sq}^3 \text{Sq}^1 + \text{Sq}^8 \text{Sq}^4) \\ &\quad + (\text{Sq}^7 \text{Sq}^1 + \text{Sq}^5 \text{Sq}^2 \text{Sq}^1 + \text{Sq}^6 \text{Sq}^2 + \text{Sq}^8) \\ &= \text{Sq}^{32} \text{Sq}^{12} \text{Sq}^4 \text{Sq}^1 + \text{Sq}^{36} \text{Sq}^8 \text{Sq}^4 \text{Sq}^1 + \text{Sq}^{38} \text{Sq}^7 \text{Sq}^3 \text{Sq}^1 \\ &\quad + \text{Sq}^{32} \text{Sq}^{16} \text{Sq}^1 + \text{Sq}^{38} \text{Sq}^8 \text{Sq}^2 \text{Sq}^1 + \text{Sq}^{40} \text{Sq}^6 \text{Sq}^2 \text{Sq}^1 \\ &\quad + \text{Sq}^{42} \text{Sq}^4 \text{Sq}^2 \text{Sq}^1 + \text{Sq}^{32} \text{Sq}^{14} \text{Sq}^2 \text{Sq}^1 + \text{Sq}^{33} \text{Sq}^{16} \\ &\quad + \text{Sq}^{32} \text{Sq}^{15} \text{Sq}^2 + \text{Sq}^{32} \text{Sq}^{12} \text{Sq}^5 + \text{Sq}^{36} \text{Sq}^{12} \text{Sq}^1 \\ &\quad + \text{Sq}^{38} \text{Sq}^{10} \text{Sq}^1 + \text{Sq}^{40} \text{Sq}^8 \text{Sq}^1 + \text{Sq}^{44} \text{Sq}^4 \text{Sq}^1 + \text{Sq}^{46} \text{Sq}^2 \text{Sq}^1 \\ &\quad + \text{Sq}^{39} \text{Sq}^{10} + \text{Sq}^{47} \text{Sq}^2 + \text{Sq}^{48} \text{Sq}^1 + \text{Sq}^{49} \\ &\quad + \text{Sq}^{39} \text{Sq}^8 \text{Sq}^2 + \text{Sq}^{43} \text{Sq}^4 \text{Sq}^2 + \text{Sq}^{45} \text{Sq}^3 \text{Sq}^1 \\ &\quad + \text{Sq}^{36} \text{Sq}^{13} + \text{Sq}^{40} \text{Sq}^9 + \text{Sq}^{45} \text{Sq}^4 + \text{Sq}^{38} \text{Sq}^8 \text{Sq}^3 \\ &\quad + \text{Sq}^{38} \text{Sq}^9 \text{Sq}^2 + \text{Sq}^{40} \text{Sq}^7 \text{Sq}^2 + \text{Sq}^{36} \text{Sq}^9 \text{Sq}^4 \\ &= \text{Sq}^8 \text{Sq}^{12} \text{Sq}^{14} \text{Sq}^{15}. \end{aligned}$$

The computations of  $\chi(X_n)$  for  $n = 1, 2, 3$  and the stripping technique will be used for computing the images  $\chi(Z_n)$  for  $n = 1, 2, 3$ .

**Theorem 3.4** For the top element  $Z_2$  considering the  $Z$ -basis in  $\mathcal{A}_2(2)$ ,  $\chi(Z_2) = \text{Sq}^{17}\text{Sq}^5\text{Sq}^1$ .

**Proof.** By Example 3.2 and

$$\chi(Z_1) = \chi(\text{Sq}^2\text{Sq}^3\text{Sq}^1) = \chi(\text{Sq}^1)\chi(\text{Sq}^3)\chi(\text{Sq}^2) = (\text{Sq}^1)(\text{Sq}^2\text{Sq}^1)(\text{Sq}^2) = \text{Sq}^5\text{Sq}^1,$$

we have

$$\chi(Z_2) = \chi(Z_1)\chi(X_2) = (\text{Sq}^5\text{Sq}^1)(\text{Sq}^4\text{Sq}^6\text{Sq}^7) = Z_2.$$

We apply  $\text{Sq}^4$ ,  $\text{Sq}^6$ , and  $\text{Sq}^7$  respectively to the monomial  $\text{Sq}^5\text{Sq}^1$  from the right. By the relation (3) we have

$$\text{Sq}^9\text{Sq}^5 = 0. \tag{7}$$

If the monomial  $\text{Sq}^9\text{Sq}^5$  is stripped by the allowable vector (4), then the following holds:

$$\text{Sq}^5\text{Sq}^5 = \text{Sq}^9\text{Sq}^1.$$

Since  $\text{Sq}^1\text{Sq}^4 = \text{Sq}^5$  we obtain

$$\text{Sq}^5\text{Sq}^1\text{Sq}^4 = \text{Sq}^9\text{Sq}^1. \tag{8}$$

Next, we apply  $\text{Sq}^6$  from right to the above relation

$$\text{Sq}^5\text{Sq}^1\text{Sq}^4\text{Sq}^6 = \text{Sq}^9\text{Sq}^1\text{Sq}^6.$$

Note that we have the following relation

$$\text{Sq}^9\text{Sq}^1\text{Sq}^6 = \text{Sq}^{13}\text{Sq}^3 \tag{9}$$

since the relation (8) is obtained from stripping it by the allowable vector (4, 2). Hence, we obtain  $\text{Sq}^5\text{Sq}^1\text{Sq}^4\text{Sq}^6 = \text{Sq}^{13}\text{Sq}^3$ . Finally, we apply  $\text{Sq}^7$  to the above relation from right to have

$$\text{Sq}^5\text{Sq}^1\text{Sq}^4\text{Sq}^6\text{Sq}^7 = \text{Sq}^{13}\text{Sq}^3\text{Sq}^7.$$

Again the following relation  $\text{Sq}^{13}\text{Sq}^3\text{Sq}^7 = \text{Sq}^{17}\text{Sq}^5\text{Sq}^1$  holds since (9) is obtained from stripping it by the allowable vector (4, 2, 1). This follows that

$$\chi(Z_2) = \text{Sq}^5\text{Sq}^1\text{Sq}^4\text{Sq}^6\text{Sq}^7 = \text{Sq}^{17}\text{Sq}^5\text{Sq}^1 = Z_2.$$

■

**Theorem 3.5** For the top element  $Z_3$  considering the  $Z$ -basis in  $\mathcal{A}_2(3)$ ,  $\chi(Z_3) = \text{Sq}^{49}\text{Sq}^{17}\text{Sq}^5\text{Sq}^1 = Z_3$ .

**Proof.** We will compute  $\chi(Z_3)$  by using the results in Examples 3.3 and Theorem 3.4. We have that

$$\chi(Z_3) = \chi(Z_2)\chi(X_3) = (\text{Sq}^{17}\text{Sq}^5\text{Sq}^1)(\text{Sq}^8\text{Sq}^{12}\text{Sq}^{14}\text{Sq}^{15}).$$

We apply  $Sq^8$ ,  $Sq^{12}$ ,  $Sq^{14}$ , and  $Sq^{15}$  respectively to the monomial  $Sq^{17}Sq^5Sq^1$  from the right. Note that the relation

$$Sq^{17}Sq^5Sq^1Sq^8 = Sq^{25}Sq^5Sq^1 \quad (10)$$

holds since if it is stripped by the allowable vector (8), we have

$$Sq^9Sq^5Sq^1Sq^8 + Sq^{17}Sq^5Sq^1 = Sq^{17}Sq^5Sq^1$$

so  $Sq^9Sq^5Sq^1Sq^8 = 0$  by the relation (7). Then apply  $Sq^{12}$  from right to the relation (10), we have now that

$$Sq^{17}Sq^5Sq^1Sq^8Sq^{12} = Sq^{25}Sq^5Sq^1Sq^{12}.$$

Again we have the relation

$$Sq^{25}Sq^5Sq^1Sq^{12} = Sq^{33}Sq^9Sq^1. \quad (11)$$

If it is stripped by the allowable vector (8, 4), we obtain

$$Sq^{17}Sq^1Sq^1Sq^{12} + Sq^{17}Sq^5Sq^1Sq^8 = Sq^{25}Sq^5Sq^1$$

so it turns into the relation (10) since  $Sq^1Sq^1 = 0$ . Hence, we have

$$Sq^{17}Sq^5Sq^1Sq^8Sq^{12} = Sq^{33}Sq^9Sq^1.$$

Next, we apply  $Sq^{14}$  to the above relation from right to obtain the new relation

$$Sq^{17}Sq^5Sq^1Sq^8Sq^{12}Sq^{14} = Sq^{33}Sq^9Sq^1Sq^{14}.$$

Note that the relation

$$Sq^{33}Sq^9Sq^1Sq^{14} = Sq^{41}Sq^{13}Sq^3 \quad (12)$$

holds since the relation (11) can be obtained from it by stripping with the allowable vector (8, 4, 2). In that case,

$$Sq^{17}Sq^5Sq^1Sq^8Sq^{12}Sq^{14} = Sq^{41}Sq^{13}Sq^3. \quad (13)$$

Finally, we apply  $Sq^{15}$  to the above relation from right and get

$$Sq^{17}Sq^5Sq^1Sq^8Sq^{12}Sq^{14}Sq^{15} = Sq^{41}Sq^{13}Sq^3Sq^{15}. \quad (14)$$

Then we have the relation  $Sq^{41}Sq^{13}Sq^3Sq^{15} = Sq^{49}Sq^{17}Sq^5Sq^1$ , since the relation (12) is obtained from it by stripping with the allowable vector (8, 4, 2, 1). Therefore, we have

$$\chi(Z_3) = Sq^{17}Sq^5Sq^1Sq^8Sq^{12}Sq^{14}Sq^{15} = Sq^{49}Sq^{17}Sq^5Sq^1 = Z_3.$$

■

### 4. Conjectures

According to the computations and results given in the previous section, we propose the following conjectures for  $\chi(X_n)$  and  $\chi(Z_n)$ . Similar to  $Z_2$  and  $Z_3$ , the authors think that applying the certain stripping relations to the images of the top elements  $Z_n$  under  $\chi$  for  $n > 4$  is also a way to prove the following conjectures.

**Conjecture 4.1**  $\chi(Z_0) = Sq^1$  and  $\chi(Z_n) = Sq^{n2^{n+1}+1}\chi(Z_{n-1}) = Z_n$  for  $n \geq 1$ . The extended version of the recursive formula above is

$$\chi(Z_n) = Sq^{n2^{n+1}+1}Sq^{(n-1)2^n+1}Sq^{(n-2)2^{n-1}+1} \dots Sq^5Sq^1$$

for  $n \geq 0$ .

**Conjecture 4.2** For  $n \geq 1$ ,  $\chi(X_n) = X_n$ .

Next, consider the monomial

$$Q_k^n = Sq^{2^n}Sq^{2^{n-1}} \dots Sq^{2^k} \tag{15}$$

for  $n \geq k \geq 0$  [2]. Then we have the following conjectures.

**Conjecture 4.3** For  $n \geq 0$ ,

$$Q_0^n Q_1^{n+1} Sq^{2^{n+2}} = Sq^{7 \cdot 2^n} Sq^{2^{n-1}+2^n} Sq^{2^{n-2}+2^{n-1}} \dots Sq^{2^0+2^1}.$$

**Conjecture 4.4** For  $n \geq 0$ ,

$$Q_0^n Sq^{2^{n+1}} = Sq^{2^n+2^{n+1}} Q_0^{n-1}. \tag{16}$$

For instance  $Sq^2Sq^1Sq^4 = Sq^6Sq^1$  for  $n = 1$  and  $Sq^8Sq^4Sq^2Sq^1Sq^{16} = Sq^{24}Sq^4Sq^2Sq^1$  for  $n = 3$ .

### Acknowledgments

The first author is granted a fellowship by the Scientific and Technological Research Council of Turkey TUBITAK-2211 and this paper is supported by Ege University Scientific Research Council.

### References

- [1] J. F. Adams, H. R. Margolis, Modules over the Steenrod algebra, *Topology*. 10 (1971), 271-282.
- [2] D. Arnon, Monomial bases in the Steenrod algebra, *J. Pure. Appl. Algebra*. 96 (1994), 215-223.
- [3] M. G. Barrat, H. Miller, On the anti-automorphism of the Steenrod algebra, *Contemporary Math*. 12 (1981), 47-52.
- [4] D. M. Davis, The anti-automorphism of the Steenrod algebra, *Proc. Amer. Math. Soc.* 44 (1974), 235-236.
- [5] I. Karaca, I. Y. Karaca, On conjugation in the Mod-p Steenrod Algebra, *Turkish J. Math.* 24 (2001), 1-7.
- [6] I. Karaca, I. Y. Karaca, Some formulas for conjugation, *Southwest J. Pure. Appl. Math.* 2 (2001), 43-52.
- [7] J. Milnor, The Steenrod algebra and its dual, *Annals of Mathematics*. 67 (1958), 150-171.
- [8] J. P. Serre, Cohomologie module 2 des complexes d'Eilenberg-MacLane, *Commentarii Mathematici Helvetici*. 27 (1953), 198-232.
- [9] J. Silverman, Conjugation and excess in the Steenrod algebra, *Proc. Amer. Math. Soc.* 119 (1993), 657-661.
- [10] N. E. Steenrod, D. B. A. Epstein, *Cohomology operations*, *Annals of Math Studies* 50, Princeton University Press, 1962.

- [11] N. D. Turgay, A remark on the conjugation in the Steenrod algebra, *Commun. Korean. Math. Soc.* 30 (3) (2015), 269-276.
- [12] T. Vergili, I. Karaca, Examples of self-dual codes over some sub-Hopf algebras of the Steenrod algebra, *Turkish J. Math.* 47 (2017), 1313-1322.
- [13] T. Vergili, I. Karaca, A note on the new basis in the mod 2 Steenrod algebra, *J. Linear. Topological. Algebra.* 7 (2) (2018), 101-107.
- [14] R. M. W. Wood, A note on bases and relations in the Steenrod algebra, *Bull. London. Math. Soc.* 27 (1995), 380-386.
- [15] R. M. W. Wood, Problems in the Steenrod algebra, *Bull. London. Math. Soc.* 30 (1998), 449-517.