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On computing of integer positive powers for one type of tridiagonal and antitridiagonal matrices of even order

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Abstract. In this paper, firstly we derive a general expression for the mth power ($m \in \mathbb{N}$) for one type of tridiagonal matrices of even order. Secondly we present a method for computing integer powers of the antitridiagonal matrices that is corresponding with these matrices. Then, we present some examples to illustrate our results and give Maple 18 procedure in order to verify our calculations.

Keywords: Tridiagonal matrices, eigenvalues, eigenvector, Jordan's form, matrix powers.

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1. Introduction and preliminaries

In recent years, computing the arbitrary positive integer powers of tridiagonal matrices and antitridiagonal matrices have been a very popular problem for researches. Tridiagonal matrices and antitridiagonal matrices are used in different areas of science and engineering. For instance; solution of difference systems [1], the numerical solution of PDE's [6], telecommunication system analysis [6, 7], texture modeling [11], image processing and coding [17]. In these areas, the computation of the powers of these matrices are necessary. Therefore, there are a lot of studies dealing with the integer powers of these matrices using the well-known expression $A^m = TJ^mT^{-1}$ [5], where J is the Jordan's form of A and T is the transforming matrix. Rimas and others have presented several papers on computing the positive integer powers of various kinds of tridiagonal matrices

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[2–4, 8–10, 13–16]. In this paper, firstly we obtain a formula for computing the entries of positive integer powers of an n-square tridiagonal matrix of the form:

$$A = \begin{bmatrix} 0 & 2 & & & \\ 1 & 0 & 1 & 0 & \\ & \ddots & \ddots & \ddots & \\ & 0 & 1 & 0 & 1 \\ & & & 2 & 0 \end{bmatrix} \quad (1)$$

where $n = 2p$ ($p \in \mathbb{N}$). Rimas in [13] presented a formula for computing entries of the matrix (1). The formula that we obtain in this paper is simpler than the Rimas formula. We also give a method for computing positive integer powers of the antitridiagonal matrices that is correspond to matrix (1). Now, we are beginning with following definition and lemma.

Definition 1.1 The Chebyshev polynomials $T_n(x)$ of the first kind are polynomials in x of degree n defined by the following relation

$$T_n(x) = \cos n\theta, \text{when } x = \cos\theta, n = 0, 1, 2, \cdots . \quad (2)$$

The first few Chebyshev polynomials of the first kind define by

$$T_0(x) = 1, T_1(x) = x, T_2(x) = 2x^2 - 1, T_3(x) = 4x^3 - 3x. \quad (3)$$

According to the following lemma, Rimas obtained eigenvalues for matrix (1).

Lemma 1.2 [13] The eigenvalues of matrix (1) are as follows:

$$\lambda_k = -2\cos\frac{(k-1)\pi}{n-1}, k = 1, 2, \cdots, n. \quad (4)$$

Since all the eigenvalues λ_k for $k = 1, 2, \ldots, n$ are simple, for each eigenvalue λ_k corresponds single Jordan cell $J_1(\lambda_k)$ in the matrix J. Taking this into account, we write down the Jordan's form of the matrix A

$$J = \text{diag}(\lambda_1, \lambda_2, \lambda_3, \cdots, \lambda_n) \quad (5)$$

and apply the relation $\lambda_k = -\lambda_{n-k+1}, k = 1, 2, \cdots, \frac{n}{2}$, we can write

$$J = \text{diag}(\lambda_1, \lambda_2, \lambda_3, \cdots, -\lambda_3, -\lambda_2, -\lambda_1). \quad (6)$$

2. The main result and its applications

In this section, we firstly find a formula for computing (i, j)th entry of the matrix A^m for ($m \in \mathbb{N}$). Secondly we present a method for computing integer powers of the antitridiagonal matrices correspond to the matrix A.
From relation $T^{-1}AT = J$, we have

$$AT = TJ. \quad (7)$$

Here, $T = [T_1, T_2, \cdots, T_n]$ and $T_i = [T_{1i}, T_{2i}, \cdots, T_{ni}]^T$ for $i = 1, 2, \cdots, n$, then we can write

$$A[T_1, T_2, T_3, \cdots, T_n] = [T_1, T_2, T_3, \cdots, T_n]diag(\lambda_1, \lambda_2, \lambda_3, \cdots, \lambda_n). \tag{8}$$

Now, we have

$$AT_i = \lambda_i T_i, i = 1, 2, 3, \cdots, n. \tag{9}$$

It follows from (9) that

$$\begin{bmatrix} 2T_{2i} \\ T_{1i} + T_{3i} \\ \vdots \\ \\ 2T_{n-1i} \end{bmatrix} = \begin{bmatrix} \lambda_i T_{1i} \\ \lambda_i T_{2i} \\ \vdots \\ \\ \lambda_i T_{ni} \end{bmatrix}, i = 1, 2, 3, \ldots, n. \tag{10}$$

Let $T_{1i} = m_i$ for $i = 1, 2, 3, \ldots, n,$. Then, by solving the set of system (10), we find the entries of the eigenvectors of the matrix A for $i = 1, 2, \cdots, n$ as follows:

$$T_{ki} = m_i T_{k-1}(\frac{\lambda_i}{2}), k = 1, 2, 3, \ldots, n, \tag{11}$$

where T_{k-1} is Chebyshev polynomial of the first kind.
Using (11), eigenvectors of A are defined by the following expression

$$T_i = \begin{bmatrix} T_{1i} \\ T_{2i} \\ \vdots \\ T_{ni} \end{bmatrix} = m_i[T_0(\frac{\lambda_i}{2}), T_1(\frac{\lambda_i}{2}), \ldots, T_{n-1}(\frac{\lambda_i}{2})]^T, i = 1, 2, \ldots, n. \tag{12}$$

We consider $m_1 = m_2 = \cdots = m_n = 1$ arbitrarily. Now, by considering (12), the matrix T is identified as follows:

$$T = \begin{bmatrix} T_0(\frac{\lambda_1}{2}) & T_0(\frac{\lambda_2}{2}) & \ldots & T_0(\frac{\lambda_{n-1}}{2}) & T_0(\frac{\lambda_n}{2}) \\ T_1(\frac{\lambda_1}{2}) & T_1(\frac{\lambda_2}{2}) & \ldots & T_1(\frac{\lambda_{n-1}}{2}) & T_1(\frac{\lambda_n}{2}) \\ \vdots & \vdots & \ldots & \vdots & \vdots \\ T_{n-2}(\frac{\lambda_1}{2}) & T_{n-2}(\frac{\lambda_2}{2}) & \ldots & T_{n-2}(\frac{\lambda_{n-1}}{2}) & T_{n-2}(\frac{\lambda_n}{2}) \\ T_{n-1}(\frac{\lambda_1}{2}) & T_{n-1}(\frac{\lambda_2}{2}) & \ldots & T_{n-1}(\frac{\lambda_{n-1}}{2}) & T_{n-1}(\frac{\lambda_n}{2}) \end{bmatrix}. \tag{13}$$

From relation $T^{-1}AT = J$, we have

$$T^{-1}A = JT^{-1}, \tag{14}$$

here, $T^{-1} = [\tau_1, \tau_2, \tau_3, \cdots, \tau_{n-1}, \tau_n]$ and $\tau_i = [\tau_{1i}, \tau_{2i}, \tau_{3i}, \cdots, \tau_{ni}]^T$ for $i = 1, 2, \cdots, n$. Then we can write

$$[\tau_1, \tau_2, \tau_3, \cdots, \tau_{n-1}, \tau_n]A = J[\tau_1, \tau_2, \tau_3, \cdots, \tau_{n-1}, \tau_n]. \tag{15}$$

From (15) follows

$$[\tau_2, 2\tau_1+\tau_3, \tau_2+\tau_4, \cdots, \tau_{n-1}+2\tau_n, \tau_{n-1}] = [J\tau_1, J\tau_2, \cdots, J\tau_{n-1}, J\tau_n]$$

or equivalently

$$\begin{bmatrix} \tau_{12} & 2\tau_{11}+\tau_{13} & \tau_{12}+\tau_{14} & \cdots & \tau_{1n-1}+2\tau_{1n} & \tau_{1n-1} \\ \tau_{22} & 2\tau_{21}+\tau_{23} & \tau_{22}+\tau_{24} & \cdots & \tau_{2n-1}+2\tau_{2n} & \tau_{2n-1} \\ \tau_{32} & 2\tau_{31}+\tau_{33} & \tau_{32}+\tau_{34} & \cdots & \tau_{3n-1}+2\tau_{3n} & \tau_{3n-1} \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ \tau_{n-12} & 2\tau_{n-11}+\tau_{n-13} & \tau_{n-12}+\tau_{n-14} & \cdots & \tau_{n-1n-1}+2\tau_{n-1n} & \tau_{n-1n-1} \\ \tau_{n2} & 2\tau_{n1}+\tau_{n3} & \tau_{n2}+\tau_{n4} & \cdots & \tau_{nn-1}+2\tau_{nn} & \tau_{nn-1} \end{bmatrix}$$

$$= \begin{bmatrix} \lambda_1\tau_{11} & \lambda_1\tau_{12} & \lambda_1\tau_{13} & \cdots & \lambda_1\tau_{1n-1} & \lambda_1\tau_{1n} \\ \lambda_2\tau_{21} & \lambda_2\tau_{22} & \lambda_2\tau_{23} & \cdots & \lambda_2\tau_{2n-1} & \lambda_2\tau_{2n} \\ \lambda_3\tau_{31} & \lambda_3\tau_{32} & \lambda_3\tau_{33} & \cdots & \lambda_3\tau_{3n-1} & \lambda_3\tau_{3n} \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ \lambda_{n-1}\tau_{n-11} & \lambda_{n-1}\tau_{n-12} & \lambda_{n-1}\tau_{n-13} & \cdots & \lambda_{n-1}\tau_{n-1n-1} & \lambda_{n-1}\tau_{n-1n} \\ \lambda_n\tau_{n1} & \lambda_n\tau_{n2} & \lambda_n\tau_{n3} & \cdots & \lambda_n\tau_{nn-1} & \lambda_n\tau_{nn} \end{bmatrix}. \tag{16}$$

Let $\tau_{i1} = m_i$ for $i = 1, 2, \cdots, n$. Then, after solving the set of system (16), we find the matrix T^{-1} as follows:

$$T^{-1} = \begin{bmatrix} m_1T_0(\frac{\lambda_1}{2}) & 2m_1T_0(\frac{\lambda_1}{2}) & \ldots & 2m_1T_0(\frac{\lambda_1}{2}) & m_1T_0(\frac{\lambda_1}{2}) \\ 2m_2T_1(\frac{\lambda_2}{2}) & 2m_2T_1(\frac{\lambda_2}{2}) & \ldots & 2m_2T_1(\frac{\lambda_2}{2}) & 2m_2T_1(\frac{\lambda_2}{2}) \\ \vdots & \vdots & \ldots & \vdots & \vdots \\ 2m_{n-1}T_{n-2}(\frac{\lambda_{n-1}}{2}) & 2m_{n-1}T_{n-2}(\frac{\lambda_{n-1}}{2}) & \ldots & 2m_{n-1}T_{n-2}(\frac{\lambda_{n-1}}{2}) & 2m_{n-1}T_{n-2}(\frac{\lambda_{n-1}}{2}) \\ m_nT_{n-1}(\frac{\lambda_n}{2}) & 2m_nT_{n-1}(\frac{\lambda_n}{2}) & \ldots & 2m_nT_{n-1}(\frac{\lambda_n}{2}) & m_nT_{n-1}(\frac{\lambda_n}{2}) \end{bmatrix}. \tag{17}$$

It follows from (4), (13) and (17) that

$$T = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 & 1 & 1 \\ -1 & -\cos\frac{\pi}{n-1} & -\cos\frac{2\pi}{n-1} & \cdots & \cos\frac{2\pi}{n-1} & \cos\frac{\pi}{n-1} & 1 \\ 1 & \cos\frac{2\pi}{n-1} & \cos\frac{4\pi}{n-1} & \cdots & \cos\frac{4\pi}{n-1} & \cos\frac{2\pi}{n-1} & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ -1 & \cos\frac{2\pi}{n-1} & -\cos\frac{4\pi}{n-1} & \cdots & \cos\frac{4\pi}{n-1} & -\cos\frac{2\pi}{n-1} & 1 \\ 1 & -\cos\frac{\pi}{n-1} & \cos\frac{2\pi}{n-1} & \cdots & \cos\frac{2\pi}{n-1} & -\cos\frac{\pi}{n-1} & 1 \\ -1 & 1 & -1 & \cdots & 1 & -1 & 1 \end{bmatrix}, \tag{18}$$

$$T^{-1} = \begin{bmatrix} m_1 & -2m_1 & 2m_1 & \cdots & -2m_1 & 2m_1 & -m_1 \\ m_2 & -2\cos\frac{\pi}{n-1}m_2 & 2\cos\frac{2\pi}{n-1}m_2 & \cdots & 2\cos\frac{2\pi}{n-1}m_2 & -2\cos\frac{\pi}{n-1}m_2 & m_2 \\ m_3 & -2\cos\frac{2\pi}{n-1}m_3 & 2\cos\frac{4\pi}{n-1}m_3 & \cdots & -2\cos\frac{4\pi}{n-1}m_3 & 2\cos\frac{2\pi}{n-1}m_3 & -m_3 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ m_{n-2} & 2\cos\frac{2\pi}{n-1}m_{n-2} & 2\cos\frac{4\pi}{n-1}m_{n-2} & \cdots & 2\cos\frac{4\pi}{n-1}m_{n-2} & 2\cos\frac{2\pi}{n-1}m_{n-2} & m_{n-2} \\ m_{n-1} & 2\cos\frac{\pi}{n-1}m_{n-1} & 2\cos\frac{2\pi}{n-1}m_{n-1} & \cdots & -2\cos\frac{2\pi}{n-1}m_{n-1} & -2\cos\frac{\pi}{n-1}m_{n-1} & -m_{n-1} \\ m_n & 2m_n & 2m_n & \cdots & 2m_n & 2m_n & m_n \end{bmatrix}. \tag{19}$$

We have $T^{-1}T = I$, by substituting (18) and (19) in the latter equality and doing the necessary computations follows

$$m_1 = m_n = \frac{1}{2n-2} \tag{20}$$

and for $i = 2, 3, \cdots, n-1$

$$m_i(2 + 4\sum_{i=1}^{\frac{n}{2}-1} \cos^2 \frac{2i\pi}{n-1}) = 1. \tag{21}$$

It follows from $n = 2k$ that

$$m_i(2 + 4\sum_{i=1}^{k-1} \cos^2 \frac{2i\pi}{2k-1}) = 1. \tag{22}$$

From (22), we have

$$m_i(2 + 4\sum_{i=2}^{k} \cos^2 \frac{2(i-1)\pi}{2k-1}) = 1. \tag{23}$$

Now, by $\cos^2 \alpha = \frac{1}{2} + \frac{1}{2}\cos 2\alpha$, we get

$$m_i(2 + 4((k-1)\frac{1}{2} + \frac{1}{2}(\sum_{i=2}^{k} \cos \frac{2(i-1)\pi}{2k-1}))) = 1. \tag{24}$$

We can write

$$\cos \frac{2(i-1)\pi}{2k-1} = \frac{1}{2\sin\frac{\pi}{2k-1}}(\sin \frac{(2i-1)\pi}{2k-1} - \sin \frac{(2i-3)\pi}{2k-1}). \tag{25}$$

If follows from (24) and (25) that

$$m_i(2 + 4((k-1)\frac{1}{2} + \frac{1}{2}(\sum_{i=2}^{k} \frac{1}{2\sin\frac{\pi}{2k-1}}(\sin \frac{(2i-1)\pi}{2k-1} - \sin \frac{(2i-3)\pi}{2k-1})))) = 1. \tag{26}$$

Now, we convert the sum into a telescoping sum, which we can evaluate directly as

$$m_i(2 + 4((k-1)\frac{1}{2} + \frac{1}{2}(\frac{1}{2\sin\frac{\pi}{2k-1}}(\sin \frac{(2k-1)\pi}{2k-1} - \sin \frac{\pi}{2k-1})))) = 1 \tag{27}$$

Hence,

$$m_i(2 + 4((k-1)\frac{1}{2} + \frac{1}{2}(\frac{1}{2\sin\frac{\pi}{2k-1}}(0 - \sin \frac{\pi}{2k-1})))) = 1. \tag{28}$$

By a little computations, we obtain

$$m_i = \frac{1}{2k-1} = \frac{1}{n-1}, i = 2, 3, \cdots, n-1. \tag{29}$$

Now, from (19) and (29), we have

$$T^{-1} = \frac{1}{2n-2}\begin{bmatrix} 1 & -2 & 2 & \cdots & -2 & 2 & -1 \\ 2 & -4\cos\frac{\pi}{n-1} & 4\cos\frac{2\pi}{n-1} & \cdots & 4\cos\frac{2\pi}{n-1} & -4\cos\frac{\pi}{n-1} & 2 \\ 2 & -4\cos\frac{2\pi}{n-1} & 4\cos\frac{4\pi}{n-1} & \cdots & -4\cos\frac{4\pi}{n-1} & 4\cos\frac{2\pi}{n-1} & -2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 2 & 4\cos\frac{2\pi}{n-1} & 4\cos\frac{4\pi}{n-1} & \cdots & 4\cos\frac{4\pi}{n-1} & 4\cos\frac{2\pi}{n-1} & 2 \\ 2 & 4\cos\frac{\pi}{n-1} & 4\cos\frac{2\pi}{n-1} & \cdots & -4\cos\frac{2\pi}{n-1} & -4\cos\frac{\pi}{n-1} & -2 \\ 1 & 2 & 2 & \cdots & 2 & 2 & 1 \end{bmatrix}. \tag{30}$$

By substituting (5), (18) and (30) in the equality $A^m = TJ^mT^{-1}$, and doing the necessary computations, we compute the mth powers of matrix A of even order. (i,j)th entry of the matrix A^m can be given as:

$$[A^m]_{ij} = [TJ^mT^{-1}]_{ij} = \frac{1}{2n-2}\times$$
$$\begin{cases} c_{ij}(\sum_{k=1}^{\frac{n}{2}} m_k \cos\frac{(i-1)(k-1)\pi}{n-1}\cos\frac{(j-1)(k-1)\pi}{n-1}\lambda_k^m) & \text{if} \quad i,j = 1,2,\cdots,\frac{n}{2} \\ c_{ij}(\sum_{k=1}^{\frac{n}{2}} (-1)^{k-1} m_k \cos\frac{(n-i)(k-1)\pi}{n-1}\cos\frac{(j-1)(k-1)\pi}{n-1}\lambda_k^m) & \text{if } i = \frac{n}{2}+1, \frac{n}{2}+2, \cdots, n, j = 1,2,\cdots,\frac{n}{2} \\ c_{ij}(\sum_{k=1}^{\frac{n}{2}} (-1)^{k-1} m_k \cos\frac{(i-1)(k-1)\pi}{n-1}\cos\frac{(n-j)(k-1)\pi}{k-1}\lambda_k^m) & \text{if } i = 1,2,\cdots,\frac{n}{2}, j = \frac{n}{2}+1, \frac{n}{2}+2, \cdots, n \\ c_{ij}(\sum_{k=1}^{\frac{n}{2}} m_k \cos\frac{(n-i)(k-1)\pi}{n-1}\cos\frac{(n-j)(k-1)\pi}{n-1}\lambda_k^m) & \text{if} \quad i,j = \frac{n}{2}+1, \frac{n}{2}+2, \cdots, n \end{cases} \tag{31}$$

where

$$c_{ij} = \begin{cases} ((-1)^{i+j} + (-1)^m) \text{ if}, j = 1, n, i = 1,2,\cdots,n \\ 2((-1)^{i+j} + (-1)^m) \text{ if}, i,j = 2,3,\cdots,n-1, i = 1,2 \end{cases}, m_k = \begin{cases} 1 \text{ if}, k = 1 \\ 2 \text{ if}, k > 1 \end{cases}$$

In the continuation, we present a method for computing the positive integer powers of the antitridiagonal matrices correspond to the matrix (1) as follows:

$$B = \begin{bmatrix} & & & 2 & 0 \\ & 0 & 1 & 0 & 1 \\ & \iddots & \iddots & \iddots & \\ 1 & 0 & 1 & 0 & \\ 0 & 2 & & & \end{bmatrix}. \tag{32}$$

Let

$$J = \begin{bmatrix} & & & 1 \\ & 0 & 1 & \\ & & \iddots & \\ & 1 & 0 & \\ 1 & & & \end{bmatrix}. \tag{33}$$

Lemma 2.1 If matrices A, B and J have the form (1), (32) and (33) respectively, then

$$B = JA. \tag{34}$$

Proof. According to the definition of the multiplication of matrices, we have

$$[JA]_{i,j} = \sum_{k=1}^{n} [J]_{i,k}[A]_{k,j} = [A]_{n+1-i,j} = \begin{cases} 2 & \text{if} \quad i = 1, j = n-1, \\ 2 & \text{if} \quad i = n, j = 2, \\ 1 & \text{if} \; n+1-(i+j) = \pm 1, \\ 0 & \text{if} \quad \text{otherwise.} \end{cases} = [B]_{i,j}. \tag{35}$$

■

Theorem 2.2 If the matrices A, B and J have the form (1), (32) and (33) respectively, then rth power ($r \in \mathbb{N}$) of matrix B computes as follows:
For $k = 1, 2, 3, \cdots$.

$$B^r = \begin{cases} A^r & \text{for} \quad r = 2k \\ JA^r & \text{for} \; r = 2k-1 \end{cases} \tag{36}$$

Proof. We prove this lemma by induction on k. The base case of $k = 1$ is true, because from lemma 2.1 and from that $AJ = JA$ and $J = J^{-1}$ follows

$$B = JA, B^2 = A^2. \tag{37}$$

Suppose that the result is true for $k > 1$ and consider case $k + 1$.
By the induction hypothesis we have

$$B^{2k} = A^{2k}, B^{2k-1} = JA^{2k-1}. \tag{38}$$

We show that case $k + 1$ also is true

$$B^{2k+2} = B^{2k}B^2 = A^{2k}A^2 = A^{2k+2} \tag{39}$$

and

$$B^{2k+1} = B^{1+2k} = BB^{2k} = JAA^{2k} = JA^{2k+1}. \tag{40}$$

Thus the formulas also hold for $k + 1$ and the induction arguments are completed. ■

We can compute (i, j)th entry of matrix B^r in (32) by using formula (31) and Theorem 2.2.

3. Numerical considerations

In this section, we give two examples. One of them calculates the powers of the matrix (1) and the other solves the system equations difference with coefficients in form matrix (1), to verify formula given in (31).

***Example* 3.1** Let A_1 be a 4×4 tridiagonal matrix given in (1) as in the following:

$$A_1 = \begin{bmatrix} 0 & 2 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 2 & 0 \end{bmatrix}.$$

The second, third, and fourth powers of the matrix A_1 are computed as in the following:
From (4), eigenvalues of matrix A_1 can be written for $k = 1, 2, \cdots, 4$ as:
$\lambda_k = -2\cos\frac{(k-1)\pi}{n-1}$. Namely, $\lambda_1 = -2, \lambda_2 = -1, \lambda_3 = 1$ and $\lambda_4 = 2$.
From (18) and (30) we can write the transforming matrix T and its inverse as:

$$T = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & -\cos\frac{\pi}{4-1} & \cos\frac{\pi}{4-1} & 1 \\ 1 & -\cos\frac{\pi}{4-1} & -\cos\frac{\pi}{4-1} & 1 \\ -1 & 1 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & -\frac{1}{2} & \frac{1}{2} & 1 \\ 1 & -\frac{1}{2} & -\frac{1}{2} & 1 \\ -1 & 1 & -1 & 1 \end{bmatrix}$$

and

$$T^{-1} = \frac{1}{2(4)-2} \begin{bmatrix} 1 & -2 & 2 & -1 \\ 2 & -4\cos\frac{\pi}{4-1} & -4\cos\frac{\pi}{4-1} & 2 \\ 2 & 4\cos\frac{\pi}{4-1} & -4\cos\frac{\pi}{4-1} & -2 \\ 1 & 2 & 2 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{6} & -\frac{1}{3} & \frac{1}{3} & -\frac{1}{6} \\ \frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} \\ \frac{1}{6} & \frac{1}{3} & \frac{1}{3} & \frac{1}{6} \end{bmatrix}.$$

Then we get

$$A_1^2 = TJ^2T^{-1} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & -\frac{1}{2} & \frac{1}{2} & 1 \\ 1 & -\frac{1}{2} & -\frac{1}{2} & 1 \\ -1 & 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} \frac{1}{6} & -\frac{1}{3} & \frac{1}{3} & -\frac{1}{6} \\ \frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} \\ \frac{1}{6} & \frac{1}{3} & \frac{1}{3} & \frac{1}{6} \end{bmatrix} = \begin{bmatrix} 2 & 0 & 2 & 0 \\ 0 & 3 & 0 & 1 \\ 1 & 0 & 3 & 0 \\ 0 & 2 & 0 & 2 \end{bmatrix},$$

$$A_1^3 = TJ^3T^{-1} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & -\frac{1}{2} & \frac{1}{2} & 1 \\ 1 & -\frac{1}{2} & -\frac{1}{2} & 1 \\ -1 & 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} -8 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 8 \end{bmatrix} \begin{bmatrix} \frac{1}{6} & -\frac{1}{3} & \frac{1}{3} & -\frac{1}{6} \\ \frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} \\ \frac{1}{6} & \frac{1}{3} & \frac{1}{3} & \frac{1}{6} \end{bmatrix} = \begin{bmatrix} 0 & 6 & 0 & 2 \\ 3 & 0 & 5 & 0 \\ 0 & 5 & 0 & 3 \\ 2 & 0 & 6 & 0 \end{bmatrix},$$

$$A_1^4 = TJ^4T^{-1} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & -\frac{1}{2} & \frac{1}{2} & 1 \\ 1 & -\frac{1}{2} & -\frac{1}{2} & 1 \\ -1 & 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 16 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 16 \end{bmatrix} \begin{bmatrix} \frac{1}{6} & -\frac{1}{3} & \frac{1}{3} & -\frac{1}{6} \\ \frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} \\ \frac{1}{6} & \frac{1}{3} & \frac{1}{3} & \frac{1}{6} \end{bmatrix} = \begin{bmatrix} 6 & 0 & 10 & 0 \\ 0 & 11 & 0 & 5 \\ 5 & 0 & 11 & 0 \\ 0 & 10 & 0 & 6 \end{bmatrix}.$$

(i,j)th entry of the matrices A_1^2, A_1^3 and A_1^4 can be verified by the formula given in (31).

***Example* 3.2** Find all functions y_1, y_2, y_3 and y_4 such that

$$\begin{cases} y_1^{'} = 2y_2 \\ y_2^{'} = y_1 + y_3 \\ y_3^{'} = y_2 + y_4 \\ y_4^{'} = 2y_3 \end{cases}$$

with initial conditions $y_1(0) = y_2(0) = y_3(0) = y_4(0) = 1$.
We can write above example in matrix form as $y^{'} = A_1 y$ with

$$y = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \\ y_4(x) \end{bmatrix}, y^{'} = \begin{bmatrix} y_1^{'}(x) \\ y_2^{'}(x) \\ y_3^{'}(x) \\ y_4^{'}(x) \end{bmatrix}, A_1 = \begin{bmatrix} 0\,2\,0\,0 \\ 1\,0\,1\,0 \\ 0\,1\,0\,1 \\ 0\,0\,2\,0 \end{bmatrix} \text{and} y(0) = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$

The following MATLAB program calculates the solution of this system.

$$>> inits =' y_1(0) = 1, y_2(0) = 1, y_3(0) = 1, y_4(0) = 1';$$

$$[y_1, y_2, y_3, y_4] = dsolve('Dy_1 = 2y_2',' Dy_2 = y_1 + y_2',' Dy_3 = y_2 + y_4',' Dy_4 = 2y_3', inits).$$

Therefore

$$y_1 = y_2 = y_3 = y_4 = e^{2x}.$$

In generally, solution of this type system as follows

$$y(x) = e^{Ax} y(0) = (I + \sum_{k=1}^{\infty} A^k \frac{x^k}{k!}) y(0).$$

For $k = 4$ we can write approximate solution as:

$$y(x) = e^{A_1 x} y(0) \approx (I + \sum_{k=1}^{4} A_1^k \frac{x^k}{k!}) y(0) = (I + xA_1 + \frac{x^2}{2} A_1^2 + \frac{x^3}{6} A_1^3 + \frac{x^4}{24} A_1^4) y(0).$$

Thus, by using Example 3.1, we have

$$y(x) \approx (\begin{bmatrix} 1\,0\,0\,0 \\ 0\,1\,0\,0 \\ 0\,0\,1\,0 \\ 0\,0\,0\,1 \end{bmatrix} + x \begin{bmatrix} 0\,2\,0\,0 \\ 1\,0\,1\,0 \\ 0\,1\,0\,1 \\ 0\,0\,2\,0 \end{bmatrix} + \frac{x^2}{2} \begin{bmatrix} 2\,0\,2\,0 \\ 0\,3\,0\,1 \\ 1\,0\,3\,0 \\ 0\,2\,0\,2 \end{bmatrix} + \frac{x^3}{6} \begin{bmatrix} 0\,6\,0\,2 \\ 3\,0\,5\,0 \\ 0\,5\,0\,3 \\ 2\,0\,6\,0 \end{bmatrix}$$

$$+ \frac{x^4}{24} \begin{bmatrix} 6 & 0 & 10 & 0 \\ 0 & 11 & 0 & 5 \\ 5 & 0 & 11 & 0 \\ 0 & 10 & 0 & 6 \end{bmatrix}) \times \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 + 2x + 2x^2 + \frac{4}{3}x^3 + \frac{2}{3}x^4 \\ 1 + 2x + 2x^2 + \frac{4}{3}x^3 + \frac{2}{3}x^4 \\ 1 + 2x + 2x^2 + \frac{4}{3}x^3 + \frac{2}{3}x^4 \\ 1 + 2x + 2x^2 + \frac{4}{3}x^3 + \frac{2}{3}x^4 \end{bmatrix} = \begin{bmatrix} \sim y1 \\ \sim y2 \\ \sim y3 \\ \sim y4 \end{bmatrix}.$$

The graphs of the exact solution (blue) and approximate solution (red) for Example 3.2 are given in Figure 1.

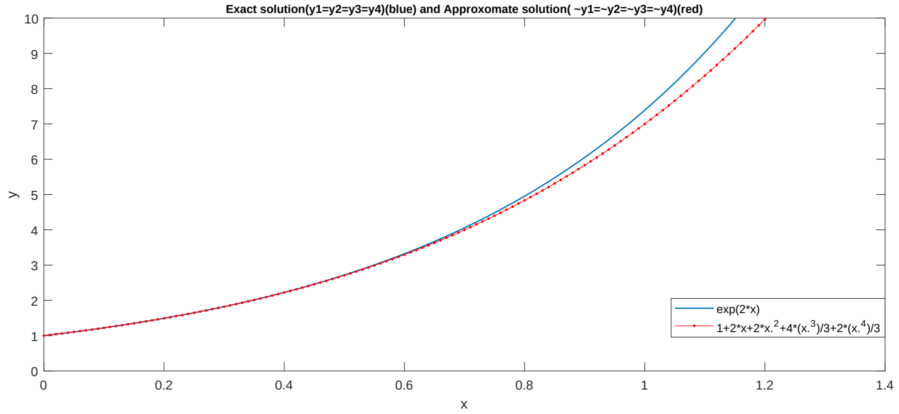

Figure 1. Comparison between exact solution (blue) and approximate solution (red) for $k = 4$.

Appendix A. Following maple 18 procedure calculates the mth power of $n-$ square tridiagonal matrix given in (1) of even order.

>restart:
with(ListTools):
Am :=**proc**(n, m)
local mm, kk, lambda, i, j, c, A, Aops;
for kk **from** 1**to** n
do
mm[kk] := 1;
if kk>1 **then** mm[kk] := 2 **end if**;
lambda[kk] := -2*cos((kk-1)*Pi/(n-1));
end do;
Aops := []:
for i **from** 1 **to** n
do
for j **from** 1**to** n
do
c[m, i, j] := 2*((-1)^(i+j)+(-1)^m):
if j = 1 **then** c[m, i, j] :=((-1)^(i+j)+(-1)^m) **end if**;
if j = n **then** c[m, i, j] :=((-1)^(i+j)+(-1)^m) **end if**;
if member(i, seq(t, t = 1 .. (1/2)*n)) **and** member(j, seq(t, t = 1 .. (1/2)*n)) =true **then**
A[m, i, j] := c[m, i, j]*(sum(mm[k]*cos((i-1)*(k-1)*Pi/(n-1))*cos((j-1)*(k-1)*Pi/(n-1))*(lambda[k])^m, k = 1 .. (1/2)*n))/(2*n-2);
end if;
if member(i, seq((1/2)*n+t,t=1..(1/2)*n)) **and** member(j, seq(t, t = 1 .. (1/2)*n))= true **then**
A[m, i, j] := c[m, i, j]*(sum((-1)^(k-1)*mm[k]*cos((n-i)*(k-1)*Pi/(n-1))*cos((j-1)*(k-1)*Pi/(n-1))*(lambda[k])^m, k = 1 .. (1/2)*n))/(2*n-2)
end if;
if member(i, seq(t, t = 1 .. (1/2)*n)) **and** member(j, seq((1/2)*n+t, t = 1 .. (1/2)*n))=

true **then**
A[m, i, j] := c[m, i, j]*(sum((-1)^(k-1)*mm[k]*cos((i-1)*(k-1)*Pi/(n-1))*cos((n-j)*(k-1)*Pi/(n-1))*(lambda[k])^m, k = 1 .. (1/2)*n))/(2*n-2)
end if;
if member(i, seq((1/2)*n+t, t = 1 .. (1/2)*n)) **and** member(j, seq((1/2)*n+t, t = 1 .. (1/2)*n)) = true **then**
A[m, i, j] := c[m, i, j]*(sum(mm[k]*cos((n-i)*(k-1)*Pi/(n-1))*cos((n-j)*(k-1)*Pi/(n-1))*(lambda[k])^m , k = 1 .. (1/2)*n))/(2*n-2) **end if**;
Aops := FlattenOnce([Aops, A[m, i, j]]);
od;
od;
print(Matrix(n, n, Aops)):
evalf(Matrix(n, n, Aops));
end proc:

Acknowledgments

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