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Continuity of some mappings on a group via semi-regular topology

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Abstract. For a given topological space $(X,\ \Im)$, there is a coarser topology on X which is called the semi-regular topology on X (generated by regularly open subsets) and it is denoted by $\Im^{\delta}$. In this paper, we study the continuity of the group operation and the inversion mapping $(\varsigma \longmapsto \varsigma^{-1})$ as regards the semi-regular topology $\Im^{\delta}$ (not necessarily with the given topology). Then we study the said mappings with the blend of the given topology $\Im$ and the semi-regular topology $\Im^{\delta}$. In the twilight of this note, we pose some questions which are noteworthy.

Keywords: $\delta-$open sets, regularly open sets, semi-regular topology, virtually topological groups, super topological groups.

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1. Introduction and preliminaries

In the following, $(X,\ \Im)$ (or simply X) denotes a topological space where $\Im$ is a nontrivial topology on X. For a given space $(X,\ \Im)$, there is a coarser topology on X which is called the semi-regular topology on X generated by the family $RO(X)$ of regularly open subsets of X and it is denoted by $\Im^{\delta}$. The space $(X,\ \Im^{\delta})$ is the semiregularization of the space $(X,\ \Im)$. For more details, refer to [8, 9].

Topological groups and paratopological groups are quite well-known in the literature of Mathematics and have a wide range of applications in different areas of Mathematics. Recall that a paratopological group is a group G with a topology such that the group

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operation of G is continuous. If in addition, the inversion map in a paratopological group is continuous, then it is called a topological group (for more details, see [1–3, 5, 6]).

In this paper, by the idea of the semi-regular topology, we first study the continuity of

(1) $((\mathcal{H},\ *),\ \Im^{\delta}) \times ((\mathcal{H},\ *),\ \Im^{\delta}) \ni (\varsigma,\ \zeta) \longmapsto \varsigma * \zeta \in ((\mathcal{H},\ *),\ \Im^{\delta})$, and

(2) $((\mathcal{H},\ *),\ \Im^{\delta}) \ni \varsigma \longmapsto \varsigma^{-1} \in ((\mathcal{H},\ *),\ \Im^{\delta})$,

and then the continuity of

(1) $((\mathcal{H},\ *),\ \Im^{\delta}) \times ((\mathcal{H},\ *),\ \Im^{\delta}) \ni (\varsigma,\ \zeta) \longmapsto \varsigma * \zeta \in ((\mathcal{H},\ *),\ \Im)$, and

(2) $((\mathcal{H},\ *),\ \Im^{\delta}) \ni \varsigma \longmapsto \varsigma^{-1} \in ((\mathcal{H},\ *),\ \Im)$,

where $(\mathcal{H},\ *)$ is a group and $\Im$ is a topology on H.

The study of the aforementioned mappings results in two new classes of topological groups, namely virtually topological groups and super topological groups. We describe various properties of both these classes of topological groups. Relationships of these spaces with topological groups are also highlighted.

In [8], regularly open sets were introduced and as a result, the concept of the semi-regularization of a given space came into existence. A subset U of a space $(X,\ \Im)$ is said to be regularly open if $U = Int(Cl(U))$ (see [8, 9]). In [9], it is shown that the family $RO(X)$ of regularly open sets in a given space X forms a base for a coarser topology $\Im^{\delta}$ on X. The space $(X,\ \Im^{\delta})$ is called the semiregularization of the space $(X,\ \Im)$.

A subset U of X is called $\delta-$open if for each $x \in U$, there exists a regularly open subset $\mathcal{P}$ of X such that $x \in \mathcal{P} \subseteq U$. Moreover, $U \subseteq X$ is regularly closed (resp. $\delta-$closed) if and only if its complement is regularly open (resp. $\delta-$open).

Let A be a subset of a space X. The $\delta-$closure of A is the intersection of all $\delta-$closed subsets of X containing A, and it is denoted by $Cl_{\delta}(A)$. The $\delta-$interior of A is the union of all $\delta-$open subsets of X that are contained in A, and it is denoted by $Int_{\delta}(A)$ (for more details, see [9]).

Definition 1.1 [4] A subset U of a space X is said to be semi-open if $U \subseteq Cl(Int(U))$.

The complement of a semi-open set is semi-closed. It is well-know that the closure of a semi-open set is regularly closed and the interior of a semi-closed set is regularly open.

2. Virtually topological groups

This section familiarizes us with the notion of virtually topological groups and some of its examples. It also acquaints us with some important properties of virtually topological groups. In particular, we demonstrate that virtually topological groups are a special class of almost regular spaces.

Definition 2.1 Let $(\mathcal{H},\ *)$ be a group, and let $\Im$ be a non-trivial topology on $\mathcal{H}$ such that

(1) the group operation $((\mathcal{H},\ *),\ \Im^{\delta}) \times ((\mathcal{H},\ *),\ \Im^{\delta}) \ni (\varsigma,\ \zeta) \longmapsto \varsigma * \zeta \in ((\mathcal{H},\ *),\ \Im^{\delta})$ and

(2) the inversion mapping $((\mathcal{H},\ *),\ \Im^{\delta}) \ni \varsigma \longmapsto \varsigma^{-1} \in ((\mathcal{H},\ *),\ \Im^{\delta})$

are continuous. Then $((\mathcal{H},\ *),\ \Im)$ is called virtually topological group.

***Example* 2.2** Let $(\mathcal{H},\ *)$ be a group of order 4. Let σ and ∇ be two non-empty disjoint subsets of $\mathcal{H}$ such that

(1) $\mathcal{H} = \sigma \cup \nabla$.

(2) $\varsigma * \sigma = \sigma$ or ∇ and $\varsigma * \nabla = \sigma$ or ∇ for each $\varsigma \in \mathcal{H}$.

(3) $\sigma * \sigma = \sigma$, $\nabla * \nabla = \sigma$ and $\sigma * \nabla = \nabla$.

Then $\mathcal{H}$ with the topology $\Im = \{\emptyset,\ \sigma,\ \nabla,\ \mathcal{H}\}$ is a virtually topological group.

***Example* 2.3** Consider the additive group $\mathcal{H} = \mathbb{Z}_2 \times \mathbb{Z}_2$ with the topology $\Im = \{\emptyset,\ \{(0,\ 0),\ (1,\ 1)\},\ \{(0,\ 1),\ (1,\ 0)\},\ \{(0,\ 1),\ (1,\ 0),\ (1,\ 1)\},\ \mathbb{Z}_2 \times \mathbb{Z}_2\}$. Then $\mathcal{H}$ with this topology $\Im$ is a virtually topological group which is not a topological group.

2.1 *Invariant topological properties*

Let $((\mathcal{H},\ *),\ \Im)$ be a virtually topological group. By π_h (resp. ℓ), we mean the mapping from $((\mathcal{H},\ *),\ \Im^\delta)$ to $((\mathcal{H},\ *),\ \Im^\delta)$ defined by $\pi_h(\varsigma) = h*\varsigma$ (resp. $\ell(\varsigma) = \varsigma^{-1}$) with $h \in H$.

Remark 1 *In light of Definition* 2.1*,* π_h *and* ℓ *are homeomorphism.*

We first give some facts about virtually topological groups. The proofs are obvious and left to the reader.

Theorem 2.4 Let $((\mathcal{H},\ *),\ \Im)$ be a virtually topological group. Then the following are equivalent:
(1) $\mathfrak{A} \in \Im^\delta$.
(2) $\varsigma * \mathfrak{A} \in \Im^\delta$ for each $\varsigma \in \mathcal{H}$.
(3) $\mathfrak{A} * \varsigma \in \Im^\delta$ for each $\varsigma \in \mathcal{H}$.
(4) $\mathfrak{A}^{-1} \in \Im^\delta$.

Theorem 2.5 Let $((\mathcal{H},\ *),\ \Im)$ be a virtually topological group. For any subset $\mathfrak{B} \subseteq \mathcal{H}$, the following hold:
(1) $Cl_\delta(\varsigma * \mathfrak{B}) = \varsigma * Cl_\delta(\mathfrak{B})$ for each $\varsigma \in \mathcal{H}$.
(2) $Int_\delta(\varsigma * \mathfrak{B}) = \varsigma * Int_\delta(\mathfrak{B})$ for each $\varsigma \in \mathcal{H}$.
(3) $Cl_\delta(\mathfrak{B}^{-1}) = [Cl_\delta(\mathfrak{B})]^{-1}$.
(4) $Int_\delta(\mathfrak{B}^{-1}) = [Int_\delta(\mathfrak{B})]^{-1}$.

Theorem 2.6 Let $((\mathcal{H},\ *),\ \Im)$ be a virtually topological group. Let $\mathfrak{B}$ be a subset of $\mathcal{H}$ satisfying one of the following conditions:
(1) $\mathfrak{B} \in RO(\mathcal{H})$.
(2) $\mathfrak{B}$ is semi-open as well as semi-closed subset of $\mathcal{H}$.
Then the following assertions are valid:
(1) $Cl(\varsigma * \mathfrak{B}) = \varsigma * Cl(\mathfrak{B})$ for each $\varsigma \in \mathcal{H}$.
(2) $Int(\varsigma * \mathfrak{B}) = \varsigma * Int(\mathfrak{B})$ for each $\varsigma \in \mathcal{H}$.
(3) $Cl(\mathfrak{B}^{-1}) = [Cl(\mathfrak{B})]^{-1}$.
(4) $Int(\mathfrak{B}^{-1}) = [Int(\mathfrak{B})]^{-1}$.

Definition 2.7 A subset A of a space X is said to be $\delta-$compact if every cover of A by $\delta-$open subsets of X has a finite subcover.

Remark 2 *Given a virtually topological group* $((\mathcal{H},\ *),\ \Im)$*, denote by* Γ *the collection of all* $\delta-$*open subsets of* $\mathcal{H}$ *containing the identity element 'e' of* $\mathcal{H}$*.*

Theorem 2.8 Let $((\mathcal{H},\ *),\ \Im)$ be a virtually topological group. Let $\mathfrak{U}$ and $\mathfrak{V}$ be disjoint subsets of $\mathcal{H}$ such that $\mathfrak{U}$ is $\delta-$compact and $\mathfrak{V}$ is $\delta-$closed. Then there exists $\sigma \in \Gamma$ satisfying $(\mathfrak{U} * \sigma) \cap (\mathfrak{V} * \sigma) = \emptyset$.

To prove Theorem 2.8, we need the following fact:

Theorem 2.9 Let $((\mathcal{H},\ *),\ \Im)$ be a virtually topological group. Then for every $\mathfrak{V} \in \Gamma$, there exists a symmetric $\sigma \in \Gamma$ (i.e., $\sigma = \sigma^{-1}$) such that $\sigma * \sigma \subseteq \mathfrak{V}$.

Proof. Straightforward. ■

Proof of Theorem 2.8. Let $\varsigma \in \mathfrak{U}$ be any element. By Theorem 2.9, there exists a symmetric $\sigma_\varsigma \in \Gamma$ such that $(\varsigma * \sigma_\varsigma * \sigma_\varsigma * \sigma_\varsigma) \cap \mathfrak{V} = \emptyset$ or

$$(\varsigma * \sigma_\varsigma * \sigma_\varsigma) \cap (\mathfrak{V} * \sigma_\varsigma) = \emptyset. \tag{1}$$

Pursuing in a similar way, we obtain a family of subsets

$$\Sigma = \{\varsigma * \sigma_\varsigma : \varsigma \in \mathfrak{U}\} \text{ where each } \sigma_\varsigma \text{ satisfies (1).} \tag{2}$$

By Theorem 2.4, $\varsigma * \sigma_\varsigma \in \Im^\delta$ for each $\varsigma \in \mathfrak{U}$. Since $\mathfrak{U}$ is $\delta-$compact, (2) has a finite subfamily $\Sigma_0 = \{\varsigma_i * \sigma_{\varsigma_i} : i = 1, 2, 3, ..., n\}$ such that $\mathfrak{U} \subseteq \bigcup\limits_{i=1}^{n} (\varsigma_i * \sigma_{\varsigma_i})$. Consider the set $\sigma = \bigcap\limits_{i=1}^{n} \sigma_{\varsigma_i}$. Then $\sigma \in \Gamma$ with $(\varsigma_i * \sigma_{\varsigma_i} * \sigma) \cap (\mathfrak{V} * \sigma) = \emptyset$. Thereby we have $(\mathfrak{U} * \sigma) \cap (\mathfrak{V} * \sigma) = \emptyset$. Proof is over.

Theorem 2.10 Let $((\mathcal{H}, *), \Im)$ be an abelian virtually topological group. Let $\mathfrak{U}$ and $\mathfrak{V}$ be disjoint subsets of $\mathcal{H}$ such that $\mathfrak{U}$ is compact and $\mathfrak{V}$ is regularly closed. Then $\mathfrak{U} * \mathfrak{V}$ is a closed subset of $\mathcal{H}$.

Before giving the proof of Theorem 2.10, let us for the purpose of vivid exposition infer some more facts about virtually topological groups.

Theorem 2.11 Let $((\mathcal{H}, *), \Im)$ be an abelian virtually topological group. Let $\mathfrak{U}$ and $\mathfrak{V}$ be disjoint subsets of $\mathcal{H}$ such that $\mathfrak{U}$ is $\delta-$compact and $\mathfrak{V}$ is $\delta-$closed. Then $\mathfrak{U} * \mathfrak{V}$ is a $\delta-$closed subset of $\mathcal{H}$.

Proof. Let $\varsigma \notin \mathfrak{U} * \mathfrak{V}$. For any $\nu \in \mathfrak{U}$, π_ν is a homeomorphism, so $\nu * \mathfrak{V}$ is $\delta-$closed. By virtue of Theorem 2.8, there exists $\sigma_\nu \in \Gamma$ such that

$$(\varsigma * \sigma_\nu) \cap (\nu * \mathfrak{V} * \sigma_\nu) = \emptyset. \tag{3}$$

Fix ς and vary ν over each element of $\mathfrak{U}$, we get the family of $\delta-$open subsets $\Sigma = \{\nu * \sigma_\nu : \sigma_\nu \in \Gamma, \ \nu \in \mathfrak{U}\}$, where each σ_ν satisfies (3) such that $\mathfrak{U} \subseteq \bigcup\{\nu * \sigma_\nu : \sigma_\nu \in \Gamma, \ \nu \in \mathfrak{U}\}$. Since $\mathfrak{U}$ is $\delta-$compact, there exists a finite subset $\mathfrak{J} \subseteq \mathfrak{U}$ such that $\mathfrak{U} \subseteq \bigcup\{\nu * \sigma_\nu : \sigma_\nu \in \Gamma, \ \nu \in \mathfrak{J}\}$. Since $(\mathcal{H}, *)$ is abelian, the set $\varsigma * \sigma$, $\sigma = \bigcap\{\sigma_\nu : \sigma_\nu \in \Gamma, \ \nu \in \mathfrak{J}\}$, does not meet $\mathfrak{U} * \mathfrak{V}$. This indicates that $\mathfrak{U} * \mathfrak{V}$ is a $\delta-$closed subset of $\mathcal{H}$. ■

Proof of Theorem 2.10. This follows by a direct application of Theorem 2.11 and Theorem 2.8.

Definition 2.12 [7] A space $(X, \Im)$ is called almost regular if for every regularly closed subset A of X and each $x \notin A$, there exist disjoint $\delta-$open subsets U and V of X such that $x \in U$ and $A \subseteq V$.

As a consequence of Theorem 2.8, we obtain the following:

Theorem 2.13 Every virtually topological group is an almost regular space.

Theorem 2.14 Let $((\mathcal{H}, *), \Im)$ be a virtually topological group. Then for every $\mathfrak{V} \in \Gamma$, there exists $\sigma \in \Gamma$ such that $Cl_\delta(\sigma) \subseteq \mathfrak{V}$.

Proof. In view of Theorem 2.8, there exists $\sigma \in \Gamma$ such that $\sigma \subseteq (\mathfrak{V}^c * \sigma)^c$ where $\mathfrak{A}^c$ denotes the complement of $\mathfrak{A} \subseteq \mathcal{H}$. Then from Theorem 2.4, it follows that $Cl_\delta(\sigma) \subseteq \mathfrak{V}$. ■

Definition 2.15 A space $(X, \Im)$ is said to be

(1) $\delta - T_0$ if for every pair of distinct points x and y of X, there exists a $\delta-$open set U in X such that either $x \in U,\ y \notin U$ or $y \in U,\ x \notin U$.

(2) $\delta - T_1$ if for every pair of distinct points x and y of X, there exist $\delta-$open sets U and V in X such that $x \in U,\ y \notin U$ and $y \in V,\ x \notin V$.

(3) $\delta - T_2$ if for every pair of distinct points x and y of X, there exist disjoint $\delta-$open sets U and V in X such that $x \in U,\ y \notin U$ and $y \in V,\ x \notin V$.

Theorem 2.16 Let $((\mathcal{H}, *), \Im)$ be a virtually topological group. Then the following are equivalent:

(1) $((\mathcal{H}, *), \Im)$ is a $\delta - T_0$ space.

(2) $((\mathcal{H}, *), \Im)$ is a $\delta - T_1$ space.

(3) $((\mathcal{H}, *), \Im)$ is a $\delta - T_2$ space.

Proof. *(1) implies (2):* Let $\varsigma \neq e$ be an element. Without loss of generality, we may assume that there exists $\mathfrak{O} \in \Gamma$ such that $\varsigma \notin \mathfrak{O}$. By Theorem 2.4, $\varsigma * \mathfrak{O} \in \Im^{\delta}$ containing ς, but $e \notin \varsigma * \mathfrak{O}$, which means that $((\mathcal{H}, *), \Im)$ is $\delta - T_1$.

(2) implies (3): Let $\varsigma \neq e$. Then there exists a set $\mathfrak{O} \in \Gamma$ such that $\varsigma \notin \mathfrak{O}$. By Theorem 2.14, there exists $\sigma \in \Gamma$ such that $\varsigma \notin Cl_{\delta}(\sigma)$. By Theorem 2.8, there exists $\nabla \in \Gamma$ such that $(\varsigma * \nabla) \cap (Cl_{\delta}(\sigma) * \nabla) = \emptyset$. Thereby the assertion follows. ■

Theorem 2.17 Let $((\mathcal{H}, *), \Im)$ be a topological group. If there exists a basis for the topology $\Im$ of $\mathcal{H}$ consisting of regularly open subsets of $\mathcal{H}$, then $((\mathcal{H}, *), \Im)$ is a virtually topological group.

Proof. The proof is obvious. ■

Theorem 2.18 Let $((\mathcal{H}, *), \Im)$ be a topological group. If there exists a subbasis for the topology $\Im$ of $\mathcal{H}$ consisting of regularly open subsets of $\mathcal{H}$, then $((\mathcal{H}, *), \Im)$ is a virtually topological group.

Proof. The proof is simple. ■

Theorem 2.19 Let $((\mathcal{H}, *), \Im)$ be a virtually topological group. For any subsets $\mathfrak{A} \subseteq \mathcal{H}$ and $\mathfrak{B} \subseteq \mathcal{H}$, $Cl_{\delta}(\mathfrak{A} * \mathfrak{B}) \supseteq Cl_{\delta}(\mathfrak{A}) * Cl_{\delta}(\mathfrak{B})$.

Proof. The proof is simple. ■

Theorem 2.20 Let $((\mathcal{H}, *), \Im)$ be a virtually topological group.

(1) If $\mathfrak{H}$ is a subgroup of $\mathcal{H}$, then $Cl_{\delta}(\mathfrak{H})$ is also subgroup of $\mathcal{H}$.

(2) If $\mathfrak{H}$ is a normal subgroup of $\mathcal{H}$, then $Cl_{\delta}(\mathfrak{H})$ is also normal subgroup of $\mathcal{H}$.

Proof. (1) Let $\varsigma,\ \zeta \in Cl_{\delta}(\mathfrak{H})$. Then $\varsigma * \zeta^{-1} \in Cl_{\delta}(\mathfrak{H}) * [Cl_{\delta}(\mathfrak{H})]^{-1}$. By dint of Theorem 2.5, $\varsigma * \zeta^{-1} \in Cl_{\delta}(\mathfrak{H}) * Cl_{\delta}(\mathfrak{H}^{-1})$. By Theorem 2.19, $\varsigma * \zeta^{-1} \in Cl_{\delta}(\mathfrak{H} * \mathfrak{H}^{-1})$. This implies that $\varsigma * \zeta^{-1} \in Cl_{\delta}(\mathfrak{H})$. This proves part (1).

(2) By above part, $Cl_{\delta}(\mathfrak{H})$ is a subgroup of $\mathcal{H}$. Now, let $\varsigma \in Cl_{\delta}(\mathfrak{H})$ and $g \in \mathcal{H}$. Then $g * \varsigma * g^{-1} \in g * Cl_{\delta}(\mathfrak{H}) * g^{-1}$. By Theorem 2.5, $g * \varsigma * g^{-1} \in Cl_{\delta}(g * \mathfrak{H}) * g^{-1} = Cl_{\delta}(g * \mathfrak{H} * g^{-1}) \subseteq Cl_{\delta}(\mathfrak{H})$. Job is done. ■

3. Super topological group

This section acquaints us with the notion of super topological groups and some of its examples. We give a necessary and sufficient condition for a topological group to be a super topological group.

Definition 3.1 A super topological group, denoted by $((\mathcal{H},\ *),\ \Im)$, is a group $(\mathcal{H},\ *)$ with a topology $\Im$ on $\mathcal{H}$ such that

(1) the group operation $((\mathcal{H},\ *),\ \Im^{\delta}) \times ((\mathcal{H},\ *),\ \Im^{\delta}) \ni (\varsigma,\ \zeta) \longmapsto \varsigma + \zeta \in ((\mathcal{H},\ *),\ \Im)$ and

(2) the inversion mapping $((\mathcal{H},\ *),\ \Im^{\delta}) \ni \varsigma \longmapsto \varsigma^{-1} \in ((\mathcal{H},\ *),\ \Im)$

are continuous.

***Example* 3.2** Let $(\mathbb{R}_+,\ .)$ be the multiplicative group of positive real numbers, and endow $\mathbb{R}_+$ with the topology induced by the usual topology on $\mathbb{R}$. Then $((\mathbb{R}_+,\ .),\ \Im)$ is a super topological group.

***Example* 3.3** Let $(\mathcal{H},\ *)$ and $\Im$ be as in Example 2.2. Then $((\mathcal{H},\ *),\ \Im)$ is a super topological group.

***Example* 3.4** Let $(\mathcal{H},\ *)$ and $\Im$ be as in Example 2.3. Then $((\mathcal{H},\ *),\ \Im)$ is not a super topological group.

It is evident from the definition that every super topological group is a topological group as well as a virtually topological group. The converse is not always true. In this connection, we obtain the following facts:

Theorem 3.5 Let $((\mathcal{H},\ *),\ \Im)$ be a topological group. Then $((\mathcal{H},\ *),\ \Im)$ is a super topological group if and only if $\Im = \Im^{\delta}$.

Proof. If $\Im = \Im^{\delta}$, then there is nothing to prove. Conversely, let $\mathfrak{O}$ be an open subset of $\mathcal{H}$. Then, the preimage of $\mathfrak{O}$ under the mapping

$$((\mathcal{H},\ *),\ \Im^{\delta}) \times ((\mathcal{H},\ *),\ \Im^{\delta}) \longmapsto ((\mathcal{H},\ *),\ \Im)$$

is a $\delta-$open subset of $((\mathcal{H},\ *),\ \Im^{\delta}) \times ((\mathcal{H},\ *),\ \Im^{\delta})$, say $\mathfrak{U}$. Thus, for any $\varsigma \in \mathfrak{U}$, there exists a regularly open subset $\mathfrak{B} = \mathfrak{I} \times \mathfrak{J} \in RO(\mathcal{H} \times \mathcal{H})$ such that $\varsigma \in \mathfrak{B} \subseteq \mathfrak{U}$. By Theorem 2.4, it follows that the image of $\mathfrak{B}$ is a $\delta-$open subset of $\mathcal{H}$. Hence $\mathfrak{O} \in \Im^{\delta}$. ■

Theorem 3.6 Let $((\mathcal{H},\ *),\ \Im)$ be a virtually topological group. Then $((\mathcal{H},\ *),\ \Im)$ is a super topological group if and only if $\Im = \Im^{\delta}$.

Proof. Follows as the proof of Theorem 3.5. ■

Theorem 3.7 Every super topological group is a semi-regular space.

Proof. Follows from Theorem 3.5. ■

We end the paper with some open questions which are noteworthy to a healthy discussion about the interconnection between the virtually topological groups and the topological groups.

Question 1. Does there exists a virtually topological group $((\mathcal{H},\ *),\ \Im)$ which is also a topological group but $\Im \neq \Im^{\delta}$?

Question 2. Does there exists a finite topological group $((\mathcal{H},\ *),\ \Im)$ which is not a virtually topological group?

Question 3. Does there exists an almost regular topological group which is not a virtually topological group?

Question 4. Does there exists a regular virtually topological group which is not a topological group?

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