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An algebraic perspective on neutrosophic sets: fields and linear spaces

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Abstract. In this work, we intend to introduce and study another algebraic structure of single-valued neutrosophic sets called neutrosophic field as a continuation of our investigations on neutrosophic algebraic structures. For this goal, we define the concept of neutrosophic fields and observe some of their basic characteristics and properties. Then we give the definition of a neutrosophic linear space over the proposed neutrosophic field and consider its fundamental properties.

Keywords: Neutrosophic set, single valued neutrosophic set, field, homomorphism, linear space.

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1. Introduction

Neutrosopy is a branch of philosophy introduced by Smarandache in 1980. It is the basis of neutrosophic logic, neutrosophic probability, neutrosophic set and neutrosophic statistics. While neutrosophic set generalizes the fuzzy set, neutrosophic probability generalizes the classical and imprecise probability, neutrosophic statistics generalizes the classical and imprecise statistics, neutrosophic logic however generalizes fuzzy logic, intuitionistic logic, Boolean logic, multi-valued logic, paraconsistent logic and dialetheism. In the neutrosophic logic, each proposition is estimated to have the percentage of truth in a subset T, the percentage of indeterminacy in a subset I, and the percentage of falsity in a subset F. The use of neutrosophic theory becomes inevitable when a situation involving

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indeterminacy is to be modeled since fuzzy set theory is limited to modeling a situation involving uncertainty. From scientific and engineering point of view, the definition of a neutrosophic set was specified to the single valued neutrosophic set. The single valued neutrosophic set was introduced for the first time by Smarandache [13] and Wang et al. [14]. The single valued neutrosophic set is a generalization of classical set, fuzzy set, intuitionistic fuzzy set and paraconsistent set etc.

The introduction of neutrosophic theory has led to the establishment of the concept of neutrosophic algebraic structures. Kandasamy and Smarandache [8] for the first time introduced the concept of algebraic structures which has caused a paradigm shift in the study of algebraic structures. Single valued neutrosophic set is also applied to algebraic and topological structures (see [1–3, 9, 11, 12]). Çetkin and Aygün [4] proposed the definitions of neutrosophic subgroups [3] and neutrosophic subrings [4] of a given classical group and classical ring, respectively. Çetkin et al. [5] also defined the neutrosophic submodules based on single valued neutrosophic sets and discussed their elementary properties. In this study, as a continuation of our investigations on algebra of single valued neutrosophic sets, we present the notions of neutrosophic fields and neutrosophic linear spaces based on single-valued neutrosophic sets. The proposed notions here, are generalizations of the known notions in the literature such as fuzzy fields [10], fuzzy linear spaces over fuzzy fields [15], intuitionistic subfields [16], vague fields and vague vector spaces [6].

In this paper we proceed as follows: Section 2 gives a brief summary of single-valued neutrosophic sets and operations on these sets. In section 3, we propose the notion of a neutrosophic field of a given classical field with examples and discuss its equivalent characterizations. In addition, we observe its main properties. In section 4, we introduce the notion of neutrosophic linear spaces over neutrosophic fields and investigate some of their fundamentals.

2. Preliminaries

In this chapter, we give some preliminaries about single valued neutrosophic sets and set operations, which will be called neutrosophic sets, for simplicity.

Definition 2.1 [13] A neutrosophic set A on the universe of X is defined by $A = \{< x, t_A(x), i_A(x), f_A(x) >, x \in X\}$, where $t_A, i_A, f_A : X \to]^-0, 1^+[$ and $^-0 \leqslant t_A(x)+i_A(x)+ f_A(x) \leqslant 3^+$.

From philosophical point of view, the neutrosophic set takes the value from real standard or non standard subsets of $]^-0, 1^+[$. But in real life applications in scientific and engineering problems it is difficult to use neutrosophic set with value from real standard or non-standard subset of $]^-0, 1^+[$. Hence throughout this work, the following specified definition of a neutrosophic set known as single valued neutrosophic set is considered.

Definition 2.2 [14] Let X be a space of points (objects) with a generic element in X denoted by x. A single valued neutrosophic set (SVNS) A on X is characterized by truth-membership function t_A, indeterminacy-membership function i_A and falsity-membership function f_A. For each point x in $X, t_A(x), i_A(x), f_A(x) \in [0, 1]$. A neutrosophic set A can be written as $A = \sum\limits_{i=1}^{n} < t(x_i), i(x_i), f(x_i) > /x_i,\ x_i \in X$.

***Example* 2.3** [14] Assume that $X = \{x_1, x_2, x_3\}, x_1$ is capability, x_2 is trustworthiness and x_3 is price. The values of x_1, x_2 and x_3 are in $[0, 1]$. They are obtained from the questionnaire of some domain experts, their option could be a degree of "good service", a degree of indeterminacy and a degree of "poor service". A is a single valued neutrosophic set of X defined by

$$A =< 0.3, 0.4, 0.5 > /x_1+ < 0.5, 0.2, 0.3 > /x_2+ < 0.7, 0.2, 0.2 > /x_3.$$

Since the membership functions t_A, i_A and f_A are defined from X into the unit interval $[0, 1]$ as $t_A, i_A, f_A : X \to [0, 1]$, a (single valued) neutrosophic set A will be denoted by a mapping defined as $A : X \to [0, 1] \times [0, 1] \times [0, 1]$ and $A(x) = (t_A(x), i_A(x), f_A(x))$, for simplicity.

Definition 2.4 [11, 14] Let A and B be two neutrosophic sets on X. Then

(1) A is contained in B denoted by $A \subseteq B$ if and only if $A(x) \leqslant B(x)$. This means that $t_A(x) \leqslant t_B(x), i_A(x) \leqslant i_B(x)$ and $f_A(x) \geqslant f_B(x)$. Two sets A and B is called equal, i.e., $A = B$ iff $A \subseteq B$ and $B \subseteq A$.
(2) the union of A and B is denoted by $C = A \cup B$ and defined as $C(x) = A(x) \vee B(x)$, where $A(x) \vee B(x) = (t_A(x) \vee t_B(x), i_A(x) \vee i_B(x), f_A(x) \wedge f_B(x))$ for each $x \in X$. This means that $t_C(x) = \max\{t_A(x), t_B(x)\}$, $i_C(x) = \max\{i_A(x), i_B(x)\}$ and $f_C(x) = \min\{f_A(x), f_B(x)\}$.
(3) the intersection of A and B is denoted by $C = A \cap B$ and defined as $C(x) = A(x) \wedge B(x)$, where $A(x) \wedge B(x) = (t_A(x) \wedge t_B(x), i_A(x) \wedge i_B(x), f_A(x) \vee f_B(x))$ for each $x \in X$. This means that $t_C(x) = \min\{t_A(x), t_B(x)\}$, $i_C(x) = \min\{i_A(x), i_B(x)\}$ and $f_C(x) = \max\{f_A(x), f_B(x)\}$.
(4) the complement of A is denoted by A^c and defined as $A^c(x) = (f_A(x), 1-i_A(x), t_A(x))$ for each $x \in X$. Here $(A^c)^c = A$.

Proposition 2.5 [14] Let A, B and C be the neutrosophic sets on the common universe X. Then the following properties are satisfied:

(1) $A \cup B = B \cup A, A \cap B = B \cap A$.
(2) $A \cup (B \cup C) = (A \cup B) \cup C, A \cap (B \cap C) = (A \cap B) \cap C$.
(3) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C), A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.
(4) $A \cap \widetilde{\emptyset} = \widetilde{\emptyset}, A \cup \widetilde{\emptyset} = A, A \cup \widetilde{X} = \widetilde{X}, A \cap \widetilde{X} = A$, where $t_{\widetilde{\emptyset}} = i_{\widetilde{\emptyset}} = 0, f_{\widetilde{\emptyset}} = 1$ and $t_{\widetilde{X}} = i_{\widetilde{X}} = 1, f_{\widetilde{X}} = 0$.
(5) $(A \cup B)^c = A^c \cap B^c, (A \cap B)^c = A^c \cup B^c$.

Definition 2.6 [5] Let A and B be two neutrosophic sets on X and Y, respectively. Then the cartesian product of A and B which is denoted by $A \times B$ is a neutrosophic set on $X \times Y$ and it is defined as $(A \times B)(x, y) = A(x) \times B(y)$, where $A(x) \times B(y) = (t_{A\times B}(x, y), i_{A\times B}(x, y), f_{A\times B}(x, y))$, i.e., $t_{A\times B}(x, y) = t_A(x) \wedge t_B(y)$, $i_{A\times B}(x, y) = i_A(x) \wedge i_B(y)$ and $f_{A\times B}(x, y) = f_A(x) \vee f_B(y)$.

Definition 2.7 [3] Let A be a neutrosophic set on X and $\alpha \in [0, 1]$. Define the α-level sets of A as follows:
$(t_A)_\alpha = \{x \in X \mid t_A(x) \geqslant \alpha\}, (i_A)_\alpha = \{x \in X \mid i_A(x) \geqslant \alpha\}$ and $(f_A)^\alpha = \{x \in X \mid f_A(x) \leqslant \alpha\}$.

Definition 2.8 [3] Let $g : X_1 \to X_2$ be a function and A and B be the neutrosophic sets on X_1 and X_2, respectively. Then

(1) the image of a neutrosophic set A is a neutrosophic set on X_2 and it is defined by follows:
$g(A)(y) = (t_{g(A)}(y), i_{g(A)}(y), f_{g(A)}(y)) = (g(t_A)(y), g(i_A)(y), g(f_A)(y))$ for all $y \in X_2$,

where $g(t_A)(y) = \begin{cases} \bigvee t_A(x), & \text{if } x \in g^{-1}(y); \\ 0, & \text{otherwise} \end{cases}$,

$$g(i_A)(y) = \begin{cases} \bigvee i_A(x), & \text{if } x \in g^{-1}(y); \\ 0, & \text{otherwise} \end{cases} \text{ and } g(f_A)(y) = \begin{cases} \bigwedge f_A(x), & \text{if } x \in g^{-1}(y); \\ 1, & \text{otherwise.} \end{cases}$$

(2) the preimage of a neutrosophic set B is a neutrosophic set on X_1 and it is defined by $g^{-1}(B)(x) = (t_{g^{-1}(B)}(x), i_{g^{-1}(B)}(x), f_{g^{-1}(B)}(x)) = (t_B(g(x)), i_B(g(x)), f_B(g(x))) = B(g(x))$ for all $x \in X_1$.

Definition 2.9 [3] Let $(X, \cdot)$ be a classical group and A be a neutrosophic set on X. A is called a neutrosophic subgroup of X if the following conditions are satisfied:

(N1) $A(x \cdot y) \geqslant A(x) \wedge A(y)$, i.e., $t_A(x \cdot y) \geqslant t_A(x) \wedge t_A(y), i_A(x \cdot y) \geqslant i_A(x) \wedge i_A(y)$ and $f_A(x \cdot y) \leqslant f_A(x) \vee f_A(y)$, $\forall x, y \in X$

(N2) $A(x^{-1}) \geqslant A(x)$, i.e., $t_A(x^{-1}) \geqslant t_A(x), i_A(x^{-1}) \geqslant i_A(x)$ and $f_A(x^{-1}) \leqslant f_A(x)$ for all $x, y \in X$.

3. Neutrosophic fields

In this section, we introduce the concept of a neutrosophic field over a given classical field, in terms of the use of the single valued neutrosophic sets. We investigate some fundamental properties and give characterizations of neutrosophic fields. From now on let $F = (F, +, \cdot)$ be a classical field with the unit elements Θ and e of the additive operation "$+$" and the multiplicative operation "$\cdot$", respectively.

Definition 3.1 Let F be a field and A be a neutrosophic set on F. Then A is called a neutrosophic field of F if the following conditions are satisfied:

(NF1) $A(x + y) \geqslant A(x) \wedge A(y)$ for all $x,\ y \in F$ i.e., $t_A(x + y) \geqslant t_A(x) \wedge t_A(y)$, $i_A(x + y) \geqslant i_A(x) \wedge i_A(y)$ and $f_A(x + y) \leqslant f_A(x) \vee f_A(y)$.

(NF2) $A(-x) \geqslant A(x)$ for all $x \in F$ i.e., $t_A(-x) \geqslant t_A(x)$, $i_A(-x) \geqslant i_A(x)$ and $f_A(-x) \leqslant f_A(x)$.

(NF3) $A(xy) \geqslant A(x) \wedge A(y)$ for all $x,\ y \in F$, i.e., $t_A(xy) \geqslant t_A(x) \wedge t_A(y)$, $i_A(xy) \geqslant i_A(x) \wedge i_A(y)$ and $f_A(xy) \leqslant f_A(x) \vee f_A(y)$.

(NF4) $A(x^{-1}) \geqslant A(x)$ for all $\Theta \neq x \in F$, i.e., $t_A(x^{-1}) \geqslant t_A(x)$, $i_A(x^{-1}) \geqslant i_A(x)$ and $f_A(x^{-1}) \leqslant f_A(x)$.

The collection of all neutrosophic fields of F is denoted by $NSF(F)$.

***Example* 3.2** Let $F = \mathbb{Z}_3 = \{\overline{0}, \overline{1}, \overline{2}\}$ be a field with addition modulo and multiplicative modulo operation (see [7]). A neutrosophic set A of F defined by $A = \{\langle 1, 1, 0\rangle/\overline{0}, \langle 0.8, 0.8, 0.2\rangle/\overline{1}, \langle 0.8, 0.8, 0.2\rangle/\overline{2}\}$ is a neutrosophic filed of F.

***Example* 3.3** Let $F = \mathbb{R}$ be the set of real numbers. It is known that F is a field with natural sum and natural multiplication operations (see [7]). Let us define a single-valued neutrosophic set $A = (t_A, i_A, f_A)$ as follows:

$$t_A(x) = \begin{cases} 1, & \text{if } x \in \{0,1\} \\ 0.8, & \text{if } x \in \mathbb{Q} \setminus \{0,1\} \\ 0, & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}, \quad i_A(x) = \begin{cases} 1, & \text{if } x \in \{0,1\} \\ 0.6, & \text{if } x \in \mathbb{Q} \setminus \{0,1\} \\ 0, & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \end{cases} \text{ and}$$

$$f_A(x) = \begin{cases} 0, & \text{if } x \in \{0,1\} \\ 0.3, & \text{if } x \in \mathbb{Q} \setminus \{0,1\} \\ 1, & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}.$$

Then it is easy to see that A is a neutrosophic field of F.

Theorem 3.4 Let F be a field and A be a neutrosophic set on F. Then $A \in NSF(F)$ if and only if the followings are satisfied:

(i) A is a neutrosophic additive subgroup of $(F, +)$.
(ii) A is a neutrosophic multiplicative subgroup of $(F \setminus \{\Theta\}, \cdot)$.

Proof. The proof is straightforward. ■

Theorem 3.5 Let A be a neutrosophic field of F. Then the following properties are satisfied:

(i) $A(\Theta) \geqslant A(x)$ for any $x \in F$.
(ii) $A(e) \geqslant A(x)$ for any $x \in F \setminus \{\Theta\}$.

Proof. (i) Let $x \in F$ be given. $t_A(\Theta) = t_A(x - x) \geqslant t_A(x) \wedge t_A(-x) \geqslant t_A(x) \wedge t_A(x) = t_A(x)$ and similarly, $i_A(\Theta) \geqslant i_A(x),\ f_A(\Theta) \leqslant f_A(x)$. So the desired inequality $A(\Theta) \geqslant A(x)$ is obtained.

(ii) Let $\Theta \neq x \in F$ be given. $t_A(e) = t_A(xx^{-1}) \geqslant t_A(x) \wedge t_A(x) = t_A(x)$ and similarly we have $i_A(e) \geqslant i_A(x),\ f_A(e) \leqslant f_A(x)$. Hence $A(e) \geqslant A(x)$ for any $x \in F \setminus \{\Theta\}$. ■

Theorem 3.6 A neutrosophic set A on F is a neutrosophic field of F if and only if the following conditions are satisfied:

(i) $A(x - y) \geqslant A(x) \wedge A(y)$ for each $x, y \in F$.
(ii) $A(xy^{-1}) \geqslant A(x) \wedge A(y)$ for each $x \in F,\ \Theta \neq y \in F$.

Proof. Let A be a neutrosophic field of F and $x, y \in F$.

(i) It is clear that $t_A(x - y) \geqslant t_A(x) \wedge t_A(-y) \geqslant t_A(x) \wedge t_A(y)$. Similarly, $i_A(x - y) \geqslant i_A(x) \wedge i_A(y)$ and $f_A(x - y) \leqslant f_A(x) \wedge f_A(y)$. Hence,
$$\begin{aligned} A(x-y) &= (t_A(x-y), i_A(x-y), f_A(x-y)) \\ &\geqslant (t_A(x) \wedge t_A(y), i_A(x) \wedge i_A(y), f_A(x) \vee f_A(y)) \\ &= (t_A(x), i_A(x), f_A(x)) \wedge (t_A(y), i_A(y), f_A(y)) \\ &= A(x) \wedge A(y). \end{aligned}$$
(ii) Let $x \in F$ and $\Theta \neq y \in F$ be given. It is clear that $t_A(xy^{-1}) \geqslant t_A(x) \wedge t_A(y^{-1}) \geqslant t_A(x) \wedge t_A(y)$. Similarly, we obtain $i_A(xy^{-1}) \geqslant i_A(x) \wedge i_A(y)$ and $f_A(xy^{-1}) \leqslant f_A(x) \vee f_A(y)$. So, according to similar discussion of (i), we obtain $A(xy^{-1}) \geqslant A(x) \wedge A(y)$. The other side of the proof is clear from the proof of Theorem 3.4 in [3].

■

Theorem 3.7 Let A be a neutrosophic field of F, then the followings are satisfied:

(i) If $A(x - y) = A(\Theta)$, then $A(x) = A(y)$ for $x, y \in F$.
(ii) If $A(xy^{-1}) = A(e)$, then $A(x) = A(y)$ for $\Theta \neq x, y \in F$.

Proof.

(i) Suppose that $A(x - y) = A(\Theta)$ for some $x, y \in F$. By Theorems 3.5 (i) and 3.6 (i),
$$\begin{aligned} t_A(x) &= t_A(x-y+y) \\ &\geqslant t_A(x-y) \wedge t_A(y) \\ &= t_A(\Theta) \wedge t_A(y) \\ &= t_A(y). \end{aligned} \quad \text{and} \quad \begin{aligned} t_A(y) &= t_A(x-(x-y)) \\ &\geqslant t_A(x) \wedge t_A(x-y) \\ &= t_A(x) \wedge t_A(\Theta) \\ &= t_A(x). \end{aligned}$$
Hence we get $t_A(x) = t_A(y)$. Other equalities are obtained in a similar way. Thus, $A(x) = (t_A(x), i_A(x), f_A(x)) = (t_A(y), i_A(y), f_A(y)) = A(y)$.

(ii) Let $A(xy^{-1}) = A(e)$ for some $x \in F$ and $y \in F \setminus \{\Theta\}$. By Theorems 3.5 (ii) and 3.6 (ii), $f_A(x) = f_A(xy^{-1}y) \leqslant f_A(xy^{-1}) \vee f_A(y) = f_A(e) \vee f_A(y) = f_A(y)$. Then we obtain, $f_A(x) \leqslant f_A(y)$. Now, $f_A(y) = f_A(x(xy^{-1})^{-1}) \leqslant f_A(x) \vee f_A(xy^{-1}) = f_A(x) \vee f_A(e) = f_A(x)$. Hence, we have $f_A(y) \leqslant f_A(x)$. By these inequalities we see that $f_A(y) = f_A(x)$. Similarly, $i_A(y) = i_A(x)$ and $t_A(y) = t_A(x)$. Therefore, $A(x) = A(y)$.

■

Theorem 3.8 If A and B are two neutrosophic field of F, then their intersection $A \cap B$ so is.

Proof. By Theorem 3.6, it is sufficient to show the following conditions are true.

(i) $(A \cap B)(x - y) \geqslant (A \cap B)(x) \wedge (A \cap B)(y)$.
(ii) $(A \cap B)(xy^{-1}) \geqslant (A \cap B)(x) \wedge (A \cap B)(y)$.

(i) Let $x, y \in F$ be given.

$$\begin{aligned} t_{A\cap B}(x-y) &= t_A(x-y) \wedge t_B(x-y) \\ &\geqslant (t_A(x) \wedge t_A(y)) \wedge (t_B(x) \wedge t_B(y)) \\ &= (t_A(x) \wedge t_B(x)) \wedge (t_A(y) \wedge t_B(y)) \\ &= t_{A\cap B}(x) \wedge t_{A\cap B}(y). \end{aligned}$$

Similarly, the other inequalities are satisfied. Hence, $(A \cap B)(x - y) \geqslant (A \cap B)(x) \wedge (A \cap B)(y)$.

(ii) Let $x \in F$ and $y \in F \setminus \{\Theta\}$.

$$\begin{aligned} f_{A\cap B}(x \cdot y^{-1}) &= f_A(x \cdot y^{-1}) \vee f_B(x \cdot y^{-1}) \\ &\leqslant (f_A(x) \vee f_A(y)) \vee (f_B(x) \vee f_B(y)) \\ &= (f_A(x) \vee f_B(x)) \vee (f_A(y) \vee f_B(y)) \\ &= f_{A\cap B}(x) \vee f_{A\cap B}(y). \end{aligned}$$

The other inequalities are similarly proved. Therefore, $(A \cap B)(xy^{-1}) \geqslant (A \cap B)(x) \wedge (A \cap B)(y)$. Hence, $A \cap B \in NSF(F)$.

■

Proposition 3.9 If A is a neutrosophic field of F, then the followings are satisfied:

(i) $A(-x) = A(x)$ for all $x \in F$
(ii) $A(x^{-1}) = A(x)$ for all $\Theta \neq x \in F$.

Proof. (i) Since $A \in NSF(F)$, we have $A(-x) \geqslant A(x)$ for all $x \in F$, and also we have $A(x) = A(-(-x)) \geqslant A(-x)$. Hence, (i) is satisfied.

(ii) is similarly proved, for any $\Theta \neq x \in F$. ■

Theorem 3.10 If $A \in NSF(F)$, then $A(x+y) = A(x) \wedge A(y)$ and $A(xy) = A(x) \wedge A(y)$ with $A(x) \neq A(y)$ for each $x, y \in F$.

Proof. Let $x, y \in F$. Assume that $A(x) > A(y)$, i.e., $t_A(x) > t_A(y)$, $i_A(x) > i_A(y)$, $f_A(x) < f_A(y)$. $t_A(y) = t_A(-x + x + y) \geqslant t_A(-x) \wedge t_A(x + y) \geqslant t_A(x) \wedge t_A(x + y) \geqslant t_A(x) \wedge t_A(x) \wedge t_A(y) = t_A(y)$. Therefore, $t_A(x+y) = t_A(y) = t_A(x) \wedge t_A(y)$ for all $x, y \in F$. Now, $t_A(y) = t_A(x^{-1}xy) \geqslant t_A(x^{-1}) \wedge t_A(xy) \geqslant t_A(x) \wedge t_A(xy) \geqslant t_A(x) \wedge t_A(y) = t_A(y)$. Therefore, $t_A(xy) = t_A(y) = t_A(x) \wedge t_A(y)$ for all $x, y \in F$. Similarly, the other equalities are satisfied. ■

Theorem 3.11 Let $A \in NSF(F)$. If $A(x) < A(y)$ for some $x, y \in F$, then the followings are satisfied:

(i) $A(x+y) = A(x) = A(y+x)$.
(ii) $A(xy) = A(x) = A(yx)$.

Proof. Let $A \in NSF(F)$ and $A(x) < A(y)$ for some $x, y \in F$, i.e., $t_A(x) < t_A(y), i_A(x) < i_A(y), f_A(x) > f_A(y)$. Then $t_A(x+y) \geqslant t_A(x) \wedge t_A(y) = t_A(x)$ and $t_A(x) = t_A(x+y-y) \geqslant t_A(x+y) \wedge t_A(y) = t_A(x+y)$. Therefore, $t_A(x+y) = t_A(x)$. Similarly other equalities are obtained.

(ii) $t_A(xy) \geqslant t_A(x) \wedge t_A(y) = t_A(x)$ and $t_A(x) = t_A(xyy^{-1}) \geqslant t_A(xy) \wedge t_A(y^{-1}) \geqslant t_A(xy) \wedge t_A(y) = t_A(xy)$. Therefore, $t_A(xy) = t_A(x)$. Similarly other equalities are satisfied. ■

The following proposition gives the characterizations of a neutrosophic field in terms of α-level sets.

Proposition 3.12 $A \in NSF(F)$ if and only if for all $\alpha \in [0,1]$, α-level sets $(t_A)_\alpha, (i_A)_\alpha, (f_A)^\alpha$ of A are classical subfields of F, which are not trivial subfields.

Proof. Assume that $(t_A)_\alpha, (i_A)_\alpha, (f_A)^\alpha$ have at least one element different from Θ. Let $x, y \in (t_A)_\alpha$ (similarly, $x, y \in (i_A)_\alpha, (f_A)^\alpha$). Then $t_A(x), t_A(y) \geqslant \alpha$. So, $t_A(x-y) \geqslant t_A(x) \wedge t_A(y) \geqslant \alpha$ (and similarly, $i_A(x-y) \geqslant \alpha, f_A(x-y) \leqslant \alpha$), which gives $x - y \in (t_A)_\alpha$ (and $x - y \in (i_A)_\alpha, (f_A)^\alpha$). By the similar observation, we can see that $t_A(xy^{-1}) \geqslant \alpha$ (and $i_A(xy^{-1}) \geqslant \alpha, f_A(xy^{-1}) \leqslant \alpha$) and so $xy^{-1} \in (t_A)_\alpha$ ($xy^{-1} \in (i_A)_\alpha, (f_A)^\alpha$). Hence, $(t_A)_\alpha$ (and $(i_A)_\alpha, (f_A)^\alpha$) is a classical subfield of F.

Conversely, let $(t_A)_\alpha$ be a classical subfield of F, for each $\alpha \in [0,1]$. Let $x, y \in F$. If $x = \Theta$ or $y = \Theta$, the proof is clear. Suppose that $x, y \neq \Theta$. Choose $\alpha = t_A(x) \wedge t_A(y)$ and $\beta = t_A(x)$. Thus, $x, y \in (t_A)_\alpha$ and $x \in (t_A)_\beta$. Since $(t_A)_\alpha, (t_A)_\beta$ are classical subfields of F, $x+y, xy \in (t_A)_\alpha$ and $-x, x^{-1} \in (t_A)_\beta$. This gives $t_A(x+y) \geqslant \alpha = t_A(x) \wedge t_A(y)$, $t_A(xy) \geqslant \alpha = t_A(x) \wedge t_A(y)$, $t_A(-x) \geqslant \beta = t_A(x)$ and $t_A(x^{-1}) \geqslant \beta = t_A(x)$ for all $x \in F \setminus \{\Theta\}$. The other inequalities are similarly proved. Therefore $A \in NSF(F)$. ■

Theorem 3.13 Let F_1 and F_2 be two classical fields and $g : F_1 \to F_2$ be a homomorphism of fields which preserves the additive and multiplicative operations. Let A be a neutrosophic field of F_1, then the image $g(A)$ of A, is a neutrosophic field of F_2.

Proof. Let $A \in NSF(F_1)$ and $y_1, y_2 \in F_2$. If either $g^{-1}(y_1)$ or $g^{-1}(y_2)$ is empty, then it is obvious that $g(A) \in NS(F_2)$. Suppose $g^{-1}(y_1),\ g^{-1}(y_2) \neq \emptyset$. Let $x_1, x_2 \in F_1$ such that $g(x_1) = y_1$ and $g(x_2) = y_2$. Since g is a homomorphism of fields, then

$$
\begin{aligned}
g(t_A)(y_1 - y_2) &= \bigvee_{y_1 - y_2 = g(x)} t_A(x) \geqslant t_A(x_1 - x_2), \\
g(i_A)(y_1 - y_2) &= \bigvee_{y_1 - y_2 = g(x)} i_A(x) \geqslant i_A(x_1 - x_2), \\
g(f_A)(y_1 - y_2) &= \bigwedge_{y_1 - y_2 = g(x)} f_A(x) \leqslant f_A(x_1 - x_2).
\end{aligned}
$$

By using these inequalities, we show that $g(A)(y_1 - y_2) \geqslant g(A)(y_1) \wedge g(A)(y_2)$.

$$\begin{aligned}
g(A)(y_1 - y_2) &= (g(t_A)(y_1 - y_2), g(i_A)(y_1 - y_2), g(f_A)(y_1 - y_2)) \\
&= (\bigvee_{y_1 - y_2 = g(x)} t_A(x), \bigvee_{y_1 - y_2 = g(x)} i_A(x), \bigwedge_{y_1 - y_2 = g(x)} f_A(x)) \\
&\geqslant (t_A(x_1 - x_2), i_A(x_1 - x_2), f_A(x_1 - x_2)) \\
&\geqslant (t_A(x_1) \wedge t_A(x_2), i_A(x_1) \wedge i_A(x_2), f_A(x_1) \vee f_A(x_2)) \\
&= (t_A(x_1), i_A(x_1), f_A(x_1)) \wedge (t_A(x_2), i_A(x_2), f_A(x_2)).
\end{aligned}$$

This inequality is satisfied for each $x_1, x_2 \in F_1$ with $g(x_1) = y_1$ and $g(x_2) = y_2$,. Now, it is clear that

$$\begin{aligned}
g(A)(y_1 - y_2) &\geqslant (\bigvee_{y_1 = g(x_1)} t_A(x_1), \bigvee_{y_1 = g(x_1)} i_A(x_1), \bigwedge_{y_1 = g(x_1)} f_A(x_1)) \\
&\wedge (\bigvee_{y_2 = g(x_2)} t_A(x_2), \bigvee_{y_2 = g(x_2)} i_A(x_2), \bigwedge_{y_2 = g(x_2)} f_A(x_2)) \\
&= (g(t_A)(y_1), g(i_A)(y_1), g(f_A)(y_1)) \wedge (g(t_A)(y_2), g(i_A)(y_2), g(f_A)(y_2)) \\
&= g(A)(y_1) \wedge g(A)(y_2).
\end{aligned}$$

By the similar discussion,

$$g(A)(y_1 \cdot y_2^{-1}) \geqslant g(A)(y_1) \wedge g(A)(y_2)$$

is obtained for any $y_1 \in F_2$ and $\Theta \neq y_2 \in F_2$. Hence, the claim $g(A) \in NSF(F_2)$ is true. ■

Theorem 3.14 Let F_1 and F_2 be two classical fields and $g : F_1 \to F_2$ be a homomorphism of fields. Let B be a neutrosophic field of F_2, then the preimage $g^{-1}(B)$ is a neutrosophic field of F_1.

Proof. Let $B \in NSF(F_2)$ and $x_1, x_2 \in F_1$. Then

$$\begin{aligned}
g^{-1}(B)(x_1 - x_2) &= (t_B(g(x_1 - x_2)), i_B(g(x_1 - x_2)), f_B(g(x_1 - x_2))) \\
&= (t_B(g(x_1) - g(x_2)), i_B(g(x_1) - g(x_2)), f_B(g(x_1) - g(x_2))) \\
&\geqslant (t_B(g(x_1)) \wedge t_B(g(x_2)), i_B(g(x_1)) \\
&\quad \wedge i_B(g(x_2)), f_B(g(x_1)) \vee f_B(g(x_2))) \\
&= (t_B(g(x_1)), i_B(g(x_1)), f_B(g(x_1))) \\
&\quad \wedge (t_B(g(x_2)), i_B(g(x_2)), f_B(g(x_2))) \\
&= g^{-1}(B)(x_1) \wedge g^{-1}(B)(x_2).
\end{aligned}$$

Also, for $x_1 \in F_1$ and $\Theta \neq x_2 \in F_1$, following inequality is valid:

$$\begin{aligned}
g^{-1}(B)(x_1 \cdot x_2^{-1}) &= (t_B(g(x_1 \cdot x_2^{-1})), i_B(g(x_1 \cdot x_2^{-1})), f_B(g(x_1 \cdot x_2^{-1}))) \\
&= (t_B(g(x_1) \cdot g(x_2)^{-1}), i_B(g(x_1) \cdot g(x_2)^{-1}), f_B(g(x_1) \cdot g(x_2)^{-1})) \\
&\geqslant (t_B(g(x_1)) \wedge t_B(g(x_2)), i_B(g(x_1)) \\
&\quad \wedge i_B(g(x_2)), f_B(g(x_1)) \vee f_B(g(x_2))) \\
&= (t_B(g(x_1)), i_B(g(x_1)), f_B(g(x_1))) \\
&\quad \wedge (t_B(g(x_2)), i_B(g(x_2)), f_B(g(x_2))) \\
&= g^{-1}(B)(x_1) \wedge g^{-1}(B)(x_2).
\end{aligned}$$

Hence, $g^{-1}(B)$ is a neutrosophic field of F_1 as claimed. ■

Theorem 3.15 If A and B are neutrosophic fields of F_1 and F_2, respectively, then $A \times B$ is a neutrosophic field of $F_1 \times F_2$.

Proof. Let A and B be the neutrosophic fields of F_1 and F_2, respectively. Let $x_1, y_1 \in F_1$ and $x_2, y_2 \in F_2$. Then $(x_1, x_2), (y_1, y_2) \in F_1 \times F_2$.

$$\begin{aligned} t_{A\times B}((x_1,x_2)-(y_1,y_2)) &= t_{A\times B}(x_1-y_1,x_2-y_2) \\ &= t_A(x_1-y_1)\wedge t_B(x_2-y_2) \\ &\geqslant (t_A(x_1)\wedge t_A(y_1))\wedge(t_B(x_2)\wedge t_B(y_2)) \\ &= (t_A(x_1)\wedge t_B(x_2))\wedge(t_A(y_1)\wedge t_B(y_2)) \\ &= t_{A\times B}(x_1,x_2)\wedge t_{A\times B}(y_1,y_2). \end{aligned}$$

Similarly, we obtain $i_{A\times B}((x_1,x_2)-(y_1,y_2)) \geqslant i_{A\times B}(x_1,x_2)\wedge i_{A\times B}(y_1,y_2)$ and $f_{A\times B}((x_1,x_2)-(y_1,y_2)) \leqslant f_{A\times B}(x_1,x_2)\vee f_{A\times B}(y_1,y_2)$. Hence, $(A\times B)((x_1,x_2)-(y_1,y_2)) \geqslant (A\times B)(x_1,x_2)\wedge(A\times B)(y_1,y_2)$. So, the condition (i) of Theorem 3.6 is satisfied. Now, let us show that the condition (ii) of Theorem 3.6 is true. Let $x_1\in F_1$, $\Theta\neq y_1\in F_1$ and $x_2\in F_2$, $\Theta\neq y_2\in F_2$.

$$\begin{aligned} t_{A\times B}((x_1,x_2)(y_1,y_2)^{-1}) &= t_{A\times B}(x_1y_1^{-1},x_2y_2^{-1}) \\ &= t_A(x_1y_1^{-1})\wedge t_B(x_2y_2^{-1}) \\ &\geqslant (t_A(x_1)\wedge t_A(y_1))\wedge(t_B(x_2)\wedge t_B(y_2)) \\ &= (t_A(x_1)\wedge t_B(x_2))\wedge(t_A(y_1)\wedge t_B(y_2)) \\ &= t_{A\times B}(x_1,x_2)\wedge t_{A\times B}(y_1,y_2). \end{aligned}$$

Similarly, other inequalities are satisfied. Hence, $A\times B$ is a neutrosophic field of $F_1\times F_2$. ■

4. Neutrosophic linear spaces over neutrosophic fields

In this section, we introduce the notion of a neutrosophic linear (vector) space over the neutrosophic fields as an extension of the fuzzy linear spaces given in [15] and investigate some of its elementary properties.

Definition 4.1 Let F be a classical field, V be a classical linear (vector) space over F, A be a neutrosophic field on F, and B be a neutrosophic set on V. Then B is called a neutrosophic linear space over the neutrosophic field A if the following conditions are satisfied:

(NL1) $B(x+y)\geqslant B(x)\wedge B(y)$ for all $x,\ y\in V$ i.e., $t_B(x+y)\geqslant t_B(x)\wedge t_B(y), i_B(x+y)\geqslant i_B(x)\wedge i_B(y)$ and $f_B(x+y)\leqslant f_B(x)\vee f_B(y)$;
(NL2) $B(-x)\geqslant B(x)$ for all $x\in V$, i. e., $t_B(-x)\geqslant t_B(x), i_B(-x)\geqslant i_B(x)$ and $f_B(-x)\leqslant f_B(x)$;
(NL3) $B(\lambda x)\geqslant A(\lambda)\wedge B(x)$ for all $\lambda\in F,\ x\in V$ i.e., $t_B(\lambda x)\geqslant t_A(\lambda)\wedge t_B(x), i_A(\lambda x)\geqslant i_A(\lambda)\wedge i_B(x)$ and $f_B(\lambda x)\leqslant f_A(\lambda)\vee f_B(x)$;
(NL4) $A(e)\geqslant B(\Theta)$ i.e., $t_A(e)\geqslant t_B(\Theta), i_A(e)\geqslant i_B(\Theta)$ and $f_A(e)\leqslant f_B(\Theta)$.

Proposition 4.2 Let B be a neutrosophic linear space over a neutrosophic field A. Then $B(-y)=B(y)$ for any $y\in V$.

Proof. The proof is straightforward. ■

Proposition 4.3 If B is a neutrosophic linear space over a neutrosophic field A, then the followings are satisfied:

(i) $A(\Theta)\geqslant B(\Theta)$.
(ii) $B(\Theta)\geqslant B(x)$ for $x\in V$.
(iii) $A(\Theta)\geqslant B(x)$ for $x\in V$.

Proof. It is similar to the proof of Theorem 3.5. ■

Theorem 4.4 Let $A \in NSF(F)$, V be a linear space over F and B be a neutrosophic set of V. Then B is a neutrosophic linear space over A if and only if (i) and (ii) are satisfied:

(i) $B(\lambda x+\mu y) \geqslant (A(\lambda)\wedge B(x))\wedge(A(\mu)\wedge B(y))$ for $\lambda, \mu \in F$ and $x, y \in V$, i.e., $t_B(\lambda x+\mu y) \geqslant (t_A(\lambda) \wedge t_B(x)) \wedge (t_A(\mu) \wedge t_B(y))$, $i_B(\lambda x + \mu y) \geqslant (i_A(\lambda) \wedge t_B(x)) \wedge (i_A(\mu) \wedge t_B(y))$ and $f_B(\lambda x + \mu y) \leqslant (f_A(\lambda) \vee f_B(x)) \vee (f_A(\mu) \vee f_B(y))$.

(ii) $A(e) \geqslant B(x)$ for $x \in V$, i. e., $t_A(e) \geqslant t_B(x), i_A(e) \geqslant i_B(x)$ and $f_A(e) \leqslant f_B(x)$.

Proof. (i) Let B be a neutrosophic linear space over A. Then we have $t_B(\lambda x) \geqslant t_A(\lambda) \wedge t_B(x)$, $i_B(\lambda x) \geqslant i_A(\lambda) \wedge i_B(x)$ and $f_B(\lambda x) \leqslant f_A(\lambda) \vee f_B(x)$ for all $x \in V, \lambda \in F$ and $t_B(\mu y) \geqslant t_A(\mu) \wedge t_B(y)$, $i_B(\mu y) \geqslant i_A(\mu) \wedge i_B(y)$ and $f_B(\mu y) \leqslant f_A(\mu) \vee f_B(y)$ for all $y \in V, \mu \in F$. Hence,

$$\begin{aligned} t_B(\lambda x + \mu y) &\geqslant t_B(\lambda x) + t_B(\mu y) \\ &\geqslant (t_A(\lambda) \wedge t_B(x)) \wedge (t_A(\mu) \wedge t_B(y)) \\ &= t_A(\lambda) \wedge t_A(\mu) \wedge t_B(x) \wedge t_B(y). \end{aligned}$$

and similarly, $i_B(\lambda x + \mu y) \geqslant i_A(\lambda) \wedge i_A(\mu) \wedge i_B(x) \wedge i_B(y)$ and $f_B(\lambda x + \mu y) \leqslant f_A(\lambda) \vee f_A(\mu) \vee f_B(x) \vee f_B(y)$. Therefore, $B(\lambda x + \mu y) \geqslant (A(\lambda) \wedge B(x)) \wedge (A(\lambda) \wedge B(y))$.

(ii) It is clear from the Definiiton 4.1 (NL4) and Proposition 4.3 (ii). Conversely, let the inequalities of Theorem 4.4 are hold. For $x, y \in V$,

(NL1) If $\lambda = \mu = e$, then

$$\begin{aligned} t_B(x + y) &\geqslant t_B(ex) \wedge t_B(x) \wedge t_A(e) \wedge t_B(y) \\ &= t_A(e) \wedge t_B(x) \wedge t_B(y) \\ &= t_B(x) \wedge t_B(y). \end{aligned}$$

(NL2)

$$\begin{aligned} t_B(-x) &= t_B(\theta x + (-e)x) \\ &\geqslant t_A(\Theta) \wedge t_B(x) \wedge t_A(-e) \wedge t_B(x) \\ &= t_B(x) \wedge t_B(x) \\ &= t_B(x). \end{aligned}$$

(NL3) If $\mu = \Theta$,

$$\begin{aligned} t_B(\lambda x) &= t_B(\lambda x + \Theta x) \\ &\geqslant t_A(\lambda) \wedge t_B(x) \wedge t_A(\Theta) \wedge t_B(x) \\ &= t_A(\lambda) \wedge t_B(x) \\ &= t_B(x). \end{aligned}$$

(NL4) Obvious.

Hence, B is a neutrosophic linear space over A. ∎

Proposition 4.5 If B and C are two neutrosophic linear spaces over A, then their intersection $B \cap C$ so is.

Proof. Let $x, y \in V$, $\lambda, \mu \in F$ be given.

$$\begin{aligned} t_{B\cap C}(\lambda x + \mu y) &= t_B(\lambda x + \mu y) \wedge t_C(\lambda x + \mu y) \\ &\geqslant (t_A(\lambda) \wedge t_B(x) \wedge t_A(\mu) \wedge t_B(y)) \wedge (t_A(\lambda) \wedge t_C(x) \wedge t_A(\mu) \wedge t_C(y)) \\ &= (t_A(\lambda) \wedge t_B(x) \wedge t_A(\lambda) \wedge t_C(x)) \wedge (t_A(\mu) \wedge t_B(y) \wedge t_A(\mu) \wedge t_C(y)) \\ &= t_A(\lambda) \wedge t_{B\cap C}(x) \wedge t_A(\mu) \wedge t_{B\cap C}(y). \end{aligned}$$

The other inequalities are similarly satisfied. Hence,

$$(B \cap C)(\lambda x + \mu y) \geqslant A(\lambda) \wedge (B \cap C)(x) \wedge A(\mu) \wedge (B \cap C)(y).$$

∎

Theorem 4.6 Let V and Z be two linear spaces over the field F and $g : V \to Z$ be a linear transformation and $A \in NSF(F)$. If B is a neutrosophic linear space of A, then the image $g(B)$ is a neutrosophic linear space of A.

Proof. Let B be a neutrosophic linear space over A, $\lambda,\mu\in F$ and $z_1,z_2\in Z$. If neither $g^{-1}(z_1)$ or $g^{-1}(z_2)$ is empty, then it is obvious that $g(B)$ is a neutrosophic linear space over A. Suppose $g^{-1}(z_1),\ g^{-1}(z_2)\neq\emptyset$. Then $g^{-1}(\lambda z_1+\mu z_2)\neq\emptyset$. Let $v_1,v_2\in V$ such that $g(v_1)=z_1,\ g(v_2)=z_2$. Then $g(\lambda v_1+\mu v_2)=\lambda g(v_1)+\mu g(v_2)=\lambda z_1+\mu z_2$.

$$\begin{aligned}
g(t_B)(\lambda z_1+\mu z_2) &= \bigvee_{w\in g^{-1}(\lambda z_1+\mu z_2)} t_B(w)\\
&\geqslant \bigvee_{v_1\in g^{-1}(z_1),v_2\in g^{-1}(z_2)} t_B(\lambda v_1+\mu v_2)\\
&\geqslant \bigvee_{v_1\in g^{-1}(z_1),v_2\in g^{-1}(z_2)} \{t_A(\lambda)\wedge t_B(v_1)\wedge t_A(\mu)\wedge t_B(v_2)\}\\
&= \left(t_A(\lambda)\wedge \bigvee_{v_1\in g^{-1}(z_1)} t_B(v_1)\right)\wedge\left(t_A(\mu)\wedge \bigvee_{v_2\in g^{-1}(z_2)} t_B(z_2)\right)\\
&= t_A(\lambda)\wedge t_{g(B)}(z_1)\wedge t_A(\mu)\wedge t_{g(B)}(z_2))\\
&= t_A(\lambda)\wedge t_A(\mu)\wedge t_{g(B)}(z_1)\wedge t_{g(B)}(z_2)).
\end{aligned}$$

Similarly, we obtain
$i_{g(B)}(\lambda z_1+\mu z_2)\geqslant i_A(\lambda)\wedge i_A(\mu)\wedge i_{g(B)}(z_1)\wedge i_{g(B)}(z_2))$ and $f_{g(B)}(\lambda z_1+\mu z_2)\leqslant f_A(\lambda)\vee f_A(\mu)\vee f_{g(B)}(z_1)\vee f_{g(B)}(z_2))$.

By using these inequalities,

$$\begin{aligned}
g(B)(\lambda z_1+\mu z_2) &= (g(t_B)(\lambda z_1+\mu z_2), g(i_B)(\lambda z_1+\mu z_2), g(f_B)(\lambda z_1+\mu z_2))\\
&= \big(t_{g(B)}(\lambda z_1+\mu z_2), i_{g(B)}(\lambda z_1+\mu z_2), f_{g(B)}(\lambda z_1+\mu z_2)\big)\\
&\geqslant (t_A(\lambda)\wedge t_A(\mu)\wedge t_{g(B)}(z_1)\wedge t_{g(B)}(z_2),\\
&\quad i_A(\lambda)\wedge i_A(\mu)\wedge i_{g(B)}(z_1)\wedge i_{g(B)}(z_2),\\
&\quad f_A(\lambda)\vee f_A(\mu)\vee f_{g(B)}(z_1)\vee f_{g(B)}(z_2))\\
&= A(\lambda)\wedge A(\mu)\wedge g(B)(z_1)\wedge g(B)(z_2).
\end{aligned}$$

Obviously, for $z\in Z$ $t_A(e)\geqslant t_{g(B)}(z)$, $i_A(e)\geqslant i_{g(B)}(z)$ and $f_A(e)\leqslant f_{g(B)}(z)$. So, $A(e)\geqslant g(B)(z)$. Thus, $g(B)$ is a neutrosophic linear space over A. ■

Theorem 4.7 Let V and Z be two linear spaces over the field F and $g:V\to Z$ be a linear transformation and $A\in NSF(F)$. If D is a neutrosophic linear space of A, then the preimage $g^{-1}(D)$ is a neutrosophic linear space of A.

Proof. Let $x,y\in V$, $\lambda,\mu\in F$ be given.

$$\begin{aligned}
g^{-1}(D)(\lambda x+\mu y) &= t_D(g(\lambda x+\mu y)), i_D(g(\lambda x+\mu y)), f_D(g(\lambda x+\mu y))\\
&= (t_D(\lambda g(x)+\mu g(y)), i_D(\lambda g(x)+\mu g(y)), f_D(\lambda g(x)+\mu g(y)))\\
&\geqslant (t_A(\lambda)\wedge t_D(g(x))\wedge t_A(\mu)\wedge t_D(g(y)),\\
&\quad i_A(\lambda)\wedge i_D(g(x))\wedge i_A(\mu)\wedge i_D(g(y)),\\
&\quad f_A(\lambda)\vee f_D(g(x))\vee f_A(\mu)\vee f_D(g(y)))\\
&= (t_A(\lambda), i_A(\lambda), f_A(\lambda))\wedge(t_D(g(x)), i_D(g(x)), f_D(g(x)))\\
&\quad \wedge(t_A(\mu), i_A(\mu), f_A(\mu))\wedge(t_D(g(y)), i_D(g(y)), f_D(g(y)))\\
&= A(\lambda)\wedge g^{-1}(D)(x)\wedge A(\mu)\wedge g^{-1}(D)(y).
\end{aligned}$$

■

Proposition 4.8 Let $A\in NSF(F)$ and the neutrosophic sets $B_1,B_2,,...,B_n$ of the vector space $V_1,V_2,...,V_n$, respectively, be the neutrosophic linear spaces over A. Then $B_1\times B_2\times...\times B_n$ is a neutrosophic linear space over A.

Proof. Let $B=B_1\times B_2\times...\times B_n$, $x=(x_1,...,x_n)$, $y=(y_1,...,y_n)\in V_1\times V_2\times...\times V_n$

and $\lambda, \mu \in F$.

(i)

$$\begin{aligned} t_B(\lambda x + \mu y) &= t_{B_1 \times \ldots \times B_n}(\lambda x_1 + \mu y_1, ..., \lambda x_n + \mu y_n) \\ &= \bigwedge_{j=1,\ldots,n} t_{B_j}(\lambda x_j + \mu y_j) \\ &\geqslant \bigwedge_{j=1,\ldots,n} \{t_A(\lambda) \wedge t_{B_j}(x_j) \wedge t_A(\mu) \wedge t_{B_j}(y_j)\} \\ &= t_A(\lambda) \wedge \bigwedge_{j=1,\ldots,n} t_{B_j}(x_j) \wedge t_A(\mu) \bigwedge_{j=1,\ldots,n} t_{B_j}(y_j) \\ &= t_A(\lambda) \wedge t_B(x) \wedge t_A(\mu) \wedge t_B(y). \end{aligned}$$

Similarly, the other inequalities are hold.

(ii) $t_A(e) \geqslant t_{B_j}(x_j)$ for all $j = 1, ..., n$. So $t_A(e) \geqslant \bigwedge_j t_{B_j}(x_j) = t_B(x)$ for all $x \in V_1 \times V_2 \times ... \times V_n$.

Hence, $B_1 \times B_2 \times ... \times B_n$ is a neutrosophic linear space over A. ■

5. Conclusion

In this paper, the notion of a neutrosophic field, which based on single-valued neutrosophic sets, over a classical field has been presented. Some of its main properties have been discussed and its equivalent characterizations have been studied. At the same time, the concept of neutrosophic linear spaces over neutrosophic fields have been introduced and it is expected that several results from linear algebra and functional analysis can be extended to the notion of single-valued neutrosophic sets. Particularly, it is hoped that the concept of neutrosophic linear spaces will give rise to the notion of neutrosophic normed spaces.

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