

A class of $(2m - 1)$ -weakly amenable Banach algebras

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Abstract. Let \mathfrak{A} be a Banach space and λ be a non-zero fixed element of \mathfrak{A}^* (dual space of \mathfrak{A}) with non-zero kernel. Defining algebra product in \mathfrak{A} as $a \cdot b = \lambda(a)b$ for $a, b \in \mathfrak{A}$, we show that \mathfrak{A} is a $(2m - 1)$ -weakly amenable Banach algebra but not $2m$ -weakly amenable for any $m \in \mathbb{N}$. Furthermore, we show the converse of the statement [2, Proposition 1.4.(ii)] “for a non-unital Banach algebra \mathfrak{A} , if \mathfrak{A} is weakly amenable then $\mathfrak{A}^\#$ is weakly amenable” does not hold.

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1. Introduction and preliminaries

Let \mathfrak{A} be a Banach algebra and X be a Banach \mathfrak{A} -bimodule. A derivation $D : \mathfrak{A} \rightarrow X$ is a linear map satisfying the equation $D(ab) = D(a) \cdot b + a \cdot D(b)$. For any $x \in X$, define $\delta_x : \mathfrak{A} \rightarrow X$ by $\delta_x(a) = a \cdot x - x \cdot a$. Then δ_x is a bounded derivation from \mathfrak{A} to X . This derivation is called inner derivation. The spaces of all bounded derivations and all inner derivations from \mathfrak{A} to X are denoted by $\mathcal{Z}^1(\mathfrak{A}, X)$ and $\mathcal{N}^1(\mathfrak{A}, X)$, respectively. Then the first (topological) cohomology group of \mathfrak{A} with coefficients in X is the quotient space $\mathcal{Z}^1(\mathfrak{A}, X)/\mathcal{N}^1(\mathfrak{A}, X)$ and denoted by $\mathcal{H}^1(\mathfrak{A}, X)$ (for reviewing these concepts one may see a standard text such as [1]). Much studies have been devoted to the calculation of cohomology groups $\mathcal{H}^1(\mathfrak{A}, X)$ and the higher dimensions cohomology groups $\mathcal{H}^n(\mathfrak{A}, X)$.

A Banach algebra \mathfrak{A} is called amenable if $\mathcal{H}^1(\mathfrak{A}, X^*) = (0)$ for every Banach \mathfrak{A} -bimodule X . The Banach algebra \mathfrak{A} is said to be weakly amenable if $\mathcal{H}^1(\mathfrak{A}, \mathfrak{A}^*) = (0)$.

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The notions of n -weakly amenable ($n \in \mathbb{N}$) and permanently weakly amenable were introduced by Dales, Ghahramani and Gronbæk [2]. A Banach algebra \mathfrak{A} is said to be n -weakly amenable if $\mathcal{H}^1(\mathfrak{A}, \mathfrak{A}^{(n)}) = (0)$ and is permanently weakly amenable if $\mathcal{H}^1(\mathfrak{A}, \mathfrak{A}^{(n)}) = (0)$ for each $n \in \mathbb{N}$. They considered the weakly amenable Banach subalgebra $E \hat{\otimes} E^*$ (tensor algebra of E) of $B(E)$ for a Banach space E and proved it is $(2m - 1)$ -weakly amenable but not $2m$ -weakly amenable for any $m \in \mathbb{N}$. Their work along with Zhang's work [3] were a motivation to recognize some more such Banach algebras.

In this article, we define a special product on a given Banach space such that it becomes a $(2m - 1)$ -weakly amenable Banach algebra but not $2m$ -weakly amenable for any $m \in \mathbb{N}$.

2. Main results

Lemma 2.1 Let \mathfrak{A} be a Banach space and λ be a fixed non-zero element of \mathfrak{A}^* with non-zero kernel. Define algebra product on \mathfrak{A} as $a \cdot b = \lambda(a)b$ for $a, b \in \mathfrak{A}$. Then \mathfrak{A} is a weakly amenable Banach algebra.

Proof. Clearly, \mathfrak{A} is a non-commutative Banach algebra and the module actions can be formulated as $a \cdot \Lambda = \langle a, \Lambda \rangle \lambda$ and $\Lambda \cdot a = \lambda(a)\Lambda$ for $a \in \mathfrak{A}$ and $\Lambda \in \mathfrak{A}^*$. Let $D : \mathfrak{A} \rightarrow \mathfrak{A}^*$ be a bounded derivation. Then,

$$\begin{aligned} D(ab) &= D(a) \cdot b + a \cdot D(b), \\ D(\lambda(a)b) &= \lambda(b)D(a) + \langle a, D(b) \rangle \lambda. \end{aligned} \quad (1)$$

Now, let $a = b$, then (1) implies $\langle a, D(a) \rangle = 0$, and therefrom $\langle a + b, D(a + b) \rangle = 0$, and this in turn yields $\langle a, D(b) \rangle = -\langle b, D(a) \rangle$. Since $\lambda \neq 0$ there exists $e_0 \in \mathfrak{A}$ as left identity. Indeed, $e_0 \cdot a = \lambda(e_0)a = 1a = a$ for each $a \in \mathfrak{A}$. Hence,

$$\begin{aligned} D(e_0a) &= D(e_0) \cdot a + e_0 \cdot D(a) \\ &= D(e_0) \cdot a + \langle e_0, D(a) \rangle \lambda \\ &= D(e_0) \cdot a - \langle a, D(e_0) \rangle \lambda \\ &= D(e_0) \cdot a - a \cdot D(e_0), \quad D(e_0) \in \mathfrak{A}^*. \end{aligned}$$

■

For referring to the mentioned algebra in Lemma 2.1 we call it **Scalar algebra**.

Lemma 2.2 Let \mathfrak{A} be a Scalar algebra. Then \mathfrak{A} is a left ideal in its second dual algebra \mathfrak{A}^{**} . Furthermore, \mathfrak{A} is Arens regular.

Proof. Let \mathfrak{A} be produced by $\lambda \in \mathfrak{A}^*$. We equip \mathfrak{A}^{**} with first Arens product and identify $a \in \mathfrak{A}$ with $\hat{a} \in \mathfrak{A}^{**}$, where $\hat{\cdot}$ is the canonical embedding of \mathfrak{A} into \mathfrak{A}^{**} . Hence,

$$\psi \square a = \psi \square \hat{a} = \langle \lambda, \psi \rangle \hat{a} = \langle \lambda, \psi \rangle a.$$

For Arens regularity let $\eta \in \mathfrak{A}^*$, and $\Phi, \Psi \in \mathfrak{A}^{**}$. Then, we have

first Arens product

$$\langle \eta, \Phi \square \Psi \rangle = \langle \Psi \cdot \eta, \Phi \rangle = \langle \langle \eta, \Psi \rangle \lambda, \Phi \rangle = \langle \eta, \Psi \rangle \langle \lambda, \Phi \rangle$$

second Arens product

$$\langle \eta, \Phi \diamond \Psi \rangle = \langle \eta \cdot \Phi, \Psi \rangle = \langle \langle \lambda, \Phi \rangle \eta, \Psi \rangle = \langle \lambda, \Phi \rangle \langle \eta, \Psi \rangle .$$

■

Remark 1 In the general case, \mathfrak{A} is not a right ideal in \mathfrak{A}^{**} . For this, let \mathfrak{A} be a non-reflexive Banach algebra. Choose $\psi \in \mathfrak{A}^{**} \setminus \mathfrak{A}$ and $a \in \mathfrak{A}$ for which $\lambda(a) \neq 0$. Then $a \square \psi = \hat{a} \square \psi = \lambda(a)\psi \notin \mathfrak{A}$.

Theorem 2.3 The Scalar algebra \mathfrak{A} is $(2m - 1)$ -weakly amenable.

Proof. The Scalar algebra \mathfrak{A} is weakly amenable and has left identity. It is also a left ideal in its second dual algebra, so by [3, Theorem 3] it is $(2m - 1)$ -weakly amenable. ■

In continuation, we show that every Scalar algebra is not 2-weakly amenable. Then immediately by [2, proposition 1.2] it is not $2m$ -weakly amenable for any $m \in \mathbb{N}$.

Lemma 2.4 Let \mathfrak{A} be a Scalar algebra produced by $\lambda \in \mathfrak{A}^*$. Then $I = \{\lambda\}^\perp$ is a closed two sided ideal in \mathfrak{A}^{**} with codimension one and $\mathfrak{A}^{**} \cong I \oplus \mathbb{C}$.

Proof. We equip \mathfrak{A}^{**} with the first Arens product and define $\Phi : \mathfrak{A}^{**} \rightarrow \mathbb{C}$ by $\Phi(\psi) = \langle \lambda, \psi \rangle$. It is easy to see that Φ is a non-zero character on \mathfrak{A}^{**} . Indeed, for any ψ_1 and ψ_2 in \mathfrak{A}^{**} , we have

$$\begin{aligned} \Phi(\psi_1 \square \psi_2) &= \langle \lambda, \psi_1 \square \psi_2 \rangle \\ &= \langle \psi_2 \cdot \lambda, \psi_1 \rangle \\ &= \langle \langle \lambda, \psi_2 \rangle \lambda, \psi_1 \rangle \\ &= \Phi(\psi_1) \Phi(\psi_2). \end{aligned}$$

Clearly, $\ker \Phi = \{\lambda\}^\perp$ is a closed two sided ideal in \mathfrak{A}^{**} . Hence, $\frac{\mathfrak{A}^{**}}{I} \cong \mathbb{C}$. For short exact sequence $0 \rightarrow I \xrightarrow{\iota} \mathfrak{A}^{**} \xrightarrow{\Phi} \mathbb{C} \rightarrow 0$, define $\theta : \mathbb{C} \rightarrow \mathfrak{A}^{**}$ by $\theta(z) = z\hat{e}_0$ where e_0 is a (fixed) left identity of \mathfrak{A} . Then θ is a continuous homomorphism and $\Phi \circ \theta = \iota_{\mathbb{C}}$. Hence, the extension splits strongly; that is, $\mathfrak{A}^{**} \cong I \oplus \mathbb{C}$. ■

Proposition 2.5 The Scalar algebra \mathfrak{A} is not 2-weakly amenable.

Proof. We specify dual module actions as

$$\begin{aligned} a \cdot \psi &= \hat{a} \square \psi = \lambda(a)\psi, \\ \psi \cdot a &= \psi \square \hat{a} = \langle \lambda, \psi \rangle \hat{a}, \quad a \in \mathfrak{A}, \psi \in \mathfrak{A}^{**}. \end{aligned}$$

Let $D : \mathfrak{A} \rightarrow \mathfrak{A}^{**}$ be a continuous derivation. Then, for given $a, b \in \mathfrak{A}$, we have

$$\begin{aligned} D(a \cdot b) &= D(a) \cdot b + a \cdot D(b), \\ D(\lambda(a)b) &= \langle \lambda, D(a) \rangle \hat{b} + \lambda(a)D(b). \end{aligned}$$

Consequently, $\langle \lambda, D(a) \rangle = 0$. So, $D(a) \in \{\lambda\}^\perp = I$. It is easy to see that every bounded linear operator $D : \mathfrak{A} \rightarrow I = \{\lambda\}^\perp$ is a bounded derivation. Now, define $D : \mathfrak{A} \rightarrow I$ by $D(a) = f(a)\psi$ for some non-zero $f \in \mathfrak{A}^*$ and non-zero $\psi \in I$. If D is an inner derivation, then

$$\begin{aligned} D(a) &= f(a)\psi = \delta_w(a) \quad \text{for some } w \in \mathfrak{A}^{**} \cong I \oplus \mathbb{C}e \\ f(a)\psi &= a \cdot (V + ce) - (V + ce) \cdot a \quad \text{for some } V \in I, c \in \mathbb{C} \\ &= a \cdot V - V \cdot a \\ &= \lambda(a)V - 0 \\ &= \lambda(a)V. \end{aligned}$$

Choose $f \in \mathfrak{A}^*$ such that $f(a) = 0$ but $\lambda(a) \neq 0$ for some $a \in \mathfrak{A}$. Therefore, $V = 0$ and thereof $D = 0$. This contradiction completes the proof. ■

Now, we turn our attention to the relation between weak amenability of a non-unital Banach algebra and its unitization. By [2, Proposition 1.4] if a non-unital Banach algebra is $(2n - 1)$ -weakly amenable, then so is its unitization. By a Scalar algebra, we show its converse is not true in general. In fact, we have the following proposition.

Proposition 2.6 Let \mathfrak{A} be a Scalar algebra produced by $\lambda \in \mathfrak{A}^*$ and $I = \ker \lambda$. Then $I^\# \cong \mathfrak{A}$ is weakly amenable but I is not.

Proof. $I = \ker \lambda$ is a closed two sided ideal in \mathfrak{A} and we have the short exact sequence

$$0 \rightarrow I \xrightarrow{\iota} \mathfrak{A} \xrightarrow{\lambda} \mathbb{C} \rightarrow 0.$$

Clearly, λ is a non-zero character on \mathfrak{A} . Hence, $\frac{\mathfrak{A}}{I} \cong \mathbb{C}$. Define $\theta : \mathbb{C} \rightarrow \mathfrak{A}$ by $\theta(z) = ze_0$, where e_0 is a (fixed) left identity of \mathfrak{A} . Obviously, θ is a continuous homomorphism and $\lambda \circ \theta = \iota_{\mathbb{C}}$. Therefore, the extension splits strongly; that is, $\mathfrak{A} \cong I \oplus \mathbb{C} = I^\#$. As we saw in Lemma 2.1, $I^\# (= \mathfrak{A})$ is weakly amenable. If I is weakly amenable then I^2 is dense in I , but $I^2 = (0)$, a contradiction. Thus, I is not weakly amenable. ■

Let us consider some examples.

Example 2.7 Suppose $\mathfrak{A} = \left\{ a = \begin{pmatrix} 0 & 0 \\ a_1 & a_2 \end{pmatrix}, a_1, a_2 \in \mathbb{C} \right\}$, the subspace of $M_{2 \times 2}$ normed by $\|a\| = |a_1| + |a_2|$. It is easy to see that \mathfrak{A} with respect to matrix multiplication is a non-commutative Banach algebra. Meanwhile, if we define $\lambda : \mathfrak{A} \rightarrow \mathbb{C}$ by $\lambda(a) = a_2$ then \mathfrak{A} is a Scalar algebra. \mathfrak{A} has left identity. In fact every element as $\begin{pmatrix} 0 & 0 \\ a_1 & 1 \end{pmatrix}, (a_1 \in \mathbb{C})$ is a left identity for \mathfrak{A} . But \mathfrak{A} has no right bounded approximate identity. If otherwise, let $\{e_\alpha\}$, where $e_\alpha = \begin{pmatrix} 0 & 0 \\ e_{1\alpha} & e_{2\alpha} \end{pmatrix}, e_{1\alpha}, e_{2\alpha} \in \mathbb{C}$ be a bounded net in \mathfrak{A} such that $ae_\alpha \rightarrow a, (a \in \mathfrak{A})$. Then $\|ae_\alpha - a\| = |a_2e_{1\alpha} - a_1| + |a_2||e_{2\alpha} - 1|$ leads to contradiction. Obviously, \mathfrak{A} is a Scalar algebra. Thus, by Theorem 2.3 and Proposition 2.5, \mathfrak{A} is $(2m - 1)$ -weakly amenable but not $2m$ -weakly amenable for any $m \in \mathbb{N}$. This example can be extended to higher dimensions square matrices space.

Example 2.8 Let G be a locally compact group. $M(G)$ the space of all complex regular Borel measures on G along with $\|\mu\| = |\mu|(G)$ is a Banach space. The mapping $\lambda :$

$M(G) \longrightarrow \mathbb{C}$ defined by $\lambda(\mu) = \mu(G)$ is a bounded non-zero linear functional with non-zero kernel. We define a product in $M(G)$ by $\mu \cdot \nu = \lambda(\mu)\nu = \mu(G)\nu$. With this product, $M(G)$ is a Scalar algebra. So $M(G)$ is $(2m - 1)$ -weakly amenable Banach algebra but not $2m$ -weakly amenable for any $m \in \mathbb{N}$.

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