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Fuzzy nano Z-open sets in fuzzy nano topological spaces

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Abstract. The purpose of this work is to define and investigate a new class of sets termed fuzzy nano Z-open sets and fuzzy nano Z-closed sets in fuzzy nano topological spaces, as well as their basic properties. We also talk about fuzzy nano Z-closure and Z-interior, as well as their connections to fuzzy nano topological spaces.

Keywords: $\mathcal{F}\mathfrak{Nano}os$, $\mathcal{F}\mathfrak{Nano}\mathcal{P}os$, $\mathcal{F}\mathfrak{Nano}\delta\mathcal{S}os$, $\mathcal{F}\mathfrak{Nano}\mathcal{Z}os$.

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1. Introduction and preliminaries

In 1965, Zadeh [12] made his significant theory on fuzzy sets. Later, fuzzy topology was introduced by Chang [2]. Pawlak [7] introduced rough set theory by handling vagueness and uncertainty. This can be often defined by means of topological operations, interior and closure, called approximations. In 2013, Lellis Thivagar [5] introduced an extension of rough set theory called nano topology and defined its topological spaces in terms of approximations and boundary region of a subset of a universe using an equivalence relation on it.

Saha [8] defined δ-open sets in fuzzy topological spaces, nano topological space by Pankajam et al. [6] and neutrosophic topological space by Vadivel et al. [11]. Recently,

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Thivagar et al. [4] explored a new concept of neutrosophic nano topology, intuitionistic nano topology and fuzzy nano topology.

And also Z-open sets in topological spaces by El-Magharabi and Mubarki [3], Z-open sets in nano topological spaces by Selvaraj and Balakrishna [1] and fuzzy Z-closed sets and generalized fuzzy Z-closed sets in double fuzzy topological spaces by Shiventhiradevi et al. [9, 10].

Research Gap: No investigation on some stronger and weaker forms fuzzy nano open sets such as fuzzy nano δ open set, fuzzy nano δ-semi open set, fuzzy nano pre open set and fuzzy nano Z-open sets on fuzzy nano topological space has been reported in the fuzzy literature.

In section 2, we quickly review some preliminary notions that are essential for our study. In section 3, we introduced the concepts of $\mathcal{F}\mathfrak{Nano}o$, $\mathcal{F}\mathfrak{Nano}\delta o$, $\mathcal{F}\mathfrak{Nano}\delta\mathcal{S}o$, $\mathcal{F}\mathfrak{Nano}\mathcal{P}o$ and $\mathcal{F}\mathfrak{Nano}Zo$ sets. In fuzzy nano topological spaces, we also explore fuzzy nano Z-interior and fuzzy nano Z-closure operators.

Definition 1.1 [12] A function f from X into the unit interval I is called a fuzzy set (briefly, $\mathcal{F}s$) in X.

Definition 1.2 [12] If G and H are any two fuzzy subsets (briefly, $\mathcal{F}subs$) of a set X, then

(i) $G \leqslant H$ iff $\mu_G(l) \leqslant \mu_H(l)$, $\forall\ l$ in X.
(ii) $G = H$, if $G(l) = H(l)$, $\forall\ l$ in X.
(iii) $(G \vee H)(l) = \max\{G(l), H(l)\}$, $\forall\ l$ in X.
(iv) $(G \wedge H)(l) = \min\{G(l), H(l)\}$, $\forall\ l$ in X.

Definition 1.3 [12] The complement of a $\mathcal{F}subs$ G in X, denoted by $1-G$, is the $\mathcal{F}subs$ of X defined by $1 - G(l)$, $\forall\ l$ in X.

Definition 1.4 [4] Let V be a non-empty set and R be an equivalence relation on V. Let F be a $\mathcal{F}s$ in V with the membership function μ_F. The fuzzy nano lower (upper) approximations and fuzzy nano boundary of F in the approximation (V, R) denoted by $\underline{\mathcal{F}\mathfrak{Nano}}(F), \overline{\mathcal{F}\mathfrak{Nano}}(F)$ and $B_{\mathcal{F}\mathfrak{Nano}}(F)$ are respectively defined as follows:

(i) $\underline{\mathcal{F}\mathfrak{Nano}}(F) = \Big\{\langle l, \mu_{\underline{R}(A)}(l)\rangle / y \in [l]_R, l \in V\Big\}$,
(ii) $\overline{\mathcal{F}\mathfrak{Nano}}(F) = \Big\{\langle l, \mu_{\overline{R}(A)}(l)\rangle / y \in [l]_R, l \in V\Big\}$,
(iii) $B_{\mathcal{F}\mathfrak{Nano}}(F) = \overline{\mathcal{F}\mathfrak{Nano}}(F) - \underline{\mathcal{F}\mathfrak{Nano}}(F)$,

where $\mu_{\underline{R}(A)}(l) = \bigwedge_{y\in[l]_R} \mu_A(y)$ and $\mu_{\overline{R}(A)}(l) = \bigvee_{y\in[l]_R} \mu_A(y)$.

The collection $\tau_{\mathcal{F}}(F) = \big\{0_{\mathcal{F}}, 1_{\mathcal{F}}, \underline{\mathcal{F}\mathfrak{Nano}}(F), \overline{\mathcal{F}\mathfrak{Nano}}(F), B_{\mathcal{F}\mathfrak{Nano}}(F)\big\}$ forms a topology called as fuzzy nano topology and $(V, \tau_{\mathcal{F}}(F))$ as fuzzy nano topological space (briefly, $\mathcal{F}\mathfrak{Nano}ts$). The elements of $\tau_{\mathcal{F}}(F)$ are called fuzzy nano open (briefly, $\mathcal{F}\mathfrak{Nano}o$) sets. Elements of $[\tau_{\mathcal{F}}(F)]^c$ are called fuzzy nano closed (briefly, $\mathcal{F}\mathfrak{Nano}c$) sets.

Definition 1.5 Let $(V, \tau_{\mathcal{F}}(F))$ be a $\mathcal{F}\mathfrak{Nano}ts$. Let S be a $\mathcal{F}subs$ of V. Then fuzzy nano

(i) interior of S (briefly, $\mathcal{F}\mathfrak{Nano}int(S)$) is described as $\mathcal{F}\mathfrak{Nano}int(S) = \bigvee\{C : C \leqslant S$ & C is a $\mathcal{F}\mathfrak{Nano}o$ set$\}$.
(ii) closure of S (briefly, $\mathcal{F}\mathfrak{Nano}cl(S)$) is described as $\mathcal{F}\mathfrak{Nano}cl(S) = \bigwedge\{C : S \leqslant C$ & C is a $\mathcal{F}\mathfrak{Nano}c$ set$\}$.
(iii) regular open (briefly, $\mathcal{F}\mathfrak{Nano}ro$) set if $S = \mathcal{F}\mathfrak{Nano}int(\mathcal{F}\mathfrak{Nano}cl(S))$.
(iv) regular closed (briefly, $\mathcal{F}\mathfrak{Nano}rc$) set if $S = \mathcal{F}\mathfrak{Nano}cl(\mathcal{F}\mathfrak{Nano}int(S))$.

Definition 1.6 Let $(V, \tau_{\mathcal{F}}(F))$ be a $\mathcal{F}\mathfrak{Nano}ts$. Let S be a $\mathcal{F}subs$ of V. Then fuzzy nano

(i) δ interior of S (briefly, $\mathcal{F}\mathfrak{Nano}\delta int(S)$) is described as $\mathcal{F}\mathfrak{Nano}\delta int(S) = \bigvee\{C : C \leqslant S$ & C is a $\mathcal{F}\mathfrak{Nano}ro$ set$\}$.
(ii) δ closure of S (briefly, $\mathcal{F}\mathfrak{Nano}\delta cl(S)$) is described as $\mathcal{F}\mathfrak{Nano}\delta cl(S) = \bigwedge\{C : S \leqslant C$ & C is a $\mathcal{F}\mathfrak{Nano}rc$ set$\}$.

Definition 1.7 Let $(V, \tau_{\mathcal{F}}(F))$ be a $\mathcal{F}\mathfrak{Nano}ts$. Then a $\mathcal{F}subs$ S in V is said to be fuzzy nano

(i) δ-open (briefly, $\mathcal{F}\mathfrak{Nano}\delta o$) set if $S = \mathcal{F}\mathfrak{Nano}\delta int(S)$.
(ii) δ-α-open (or) a-open (briefly, $\mathcal{F}\mathfrak{Nano}\delta\alpha o$ (or) $\mathcal{F}\mathfrak{Nano}ao$) set if $S \leqslant \mathcal{F}\mathfrak{Nano}$ $int(\mathcal{F}\mathfrak{Nano}cl(\mathcal{F}\mathfrak{Nano}\delta int(S)))$.
(iii) δ-semi open (briefly, $\mathcal{F}\mathfrak{Nano}\delta\mathcal{S}o$) set if $S \leqslant \mathcal{F}\mathfrak{Nano}cl(\mathcal{F}\mathfrak{Nano}\delta int(S))$.
(iv) semi open (briefly, $\mathcal{F}\mathfrak{Nano}\mathcal{S}o$) set if $S \leqslant \mathcal{F}\mathfrak{Nano}cl(\mathcal{F}\mathfrak{Nano}int(S))$.
(v) pre open (briefly, $\mathcal{F}\mathfrak{Nano}\mathcal{P}o$) set if $S \leqslant \mathcal{F}\mathfrak{Nano}int(\mathcal{F}\mathfrak{Nano}cl(S))$.

The complement of an $\mathcal{F}\mathfrak{Nano}\delta o$ (resp. $\mathcal{F}\mathfrak{Nano}\delta\alpha o$, $\mathcal{F}\mathfrak{Nano}\delta\mathcal{S}o$, $\mathcal{F}\mathfrak{Nano}\mathcal{S}o$ & $\mathcal{F}\mathfrak{Nano}\mathcal{P}o$) set is called a fuzzy nano δ (resp. fuzzy nano δ-α, fuzzy nano δ-semi, fuzzy nano semi & fuzzy nano pre) closed (briefly, $\mathcal{F}\mathfrak{Nano}\delta c$ (resp. $\mathcal{F}\mathfrak{Nano}\delta\alpha c$, $\mathcal{F}\mathfrak{Nano}\delta\mathcal{S}c$, $\mathcal{F}\mathfrak{Nano}\mathcal{S}c$ & $\mathcal{F}\mathfrak{Nano}\mathcal{P}c$)) in V.

Definition 1.8 Let $(V, \tau_{\mathcal{F}}(F))$ be a $\mathcal{F}\mathfrak{Nano}ts$. Let S be a $\mathcal{F}subs$ of V. Then fuzzy nano

(i) δ semi interior of S (briefly, $\mathcal{F}\mathfrak{Nano}\delta\mathcal{S}int(S)$) is described as $\mathcal{F}\mathfrak{Nano}\delta\mathcal{S}\ int(S) = \bigvee\{C : C \leqslant S$ & C is a $\mathcal{F}\mathfrak{Nano}\delta\mathcal{S}o$ set$\}$.
(ii) δ semi closure of S (briefly, $\mathcal{F}\mathfrak{Nano}\delta\mathcal{S}cl(S)$) is described as $\mathcal{F}\mathfrak{Nano}\delta\mathcal{S}cl(S) = \bigwedge\{C : S \leqslant C$ & C is a $\mathcal{F}\mathfrak{Nano}\delta\mathcal{S}c$ set$\}$.
(iii) pre interior of S (briefly, $\mathcal{F}\mathfrak{Nano}\mathcal{P}int(S)$) is described as $\mathcal{F}\mathfrak{Nano}\mathcal{P}int(S) = \bigvee\{C : C \leqslant S$ & C is a $\mathcal{F}\mathfrak{Nano}\mathcal{P}o$ set$\}$.
(iv) pre closure of S (briefly, $\mathcal{F}\mathfrak{Nano}\mathcal{P}cl(S)$) is described as $\mathcal{F}\mathfrak{Nano}\mathcal{P}cl(S) = \bigwedge\{C : S \leqslant C$ & C is a $\mathcal{F}\mathfrak{Nano}\mathcal{P}c$ set$\}$.

2. Fuzzy nano Z-open sets

Throughout this section, let $(V, \tau_{\mathcal{F}}(F))$ be a $\mathcal{F}\mathfrak{Nano}ts$ with respect to F where F is a $\mathcal{F}subs$ of V.

Definition 2.1 Let S be a $\mathcal{F}subs$ in V is said to be a fuzzy nano

(i) Z-open (briefly, $\mathcal{F}\mathfrak{Nano}Zo$) set if $S \leqslant \mathcal{F}\mathfrak{Nano}cl(\mathcal{F}\mathfrak{Nano}\delta int(S)) \vee \mathcal{F}\mathfrak{Nano}int$ $(\mathcal{F}\mathfrak{Nano}cl(S))$,
(ii) Z-closed (briefly, $\mathcal{F}\mathfrak{Nano}Zc$) set if $\mathcal{F}\mathfrak{Nano}int(\mathcal{F}\mathfrak{Nano}\delta cl(S)) \wedge \mathcal{F}\mathfrak{Nano}cl(\mathcal{F}$ $\mathfrak{Nano}int(S)) \leqslant S$.

All $\mathcal{F}\mathfrak{Nano}Zo$ (resp. $\mathcal{F}\mathfrak{Nano}Zc$) sets of a space $(V, \tau_{\mathcal{F}}(F))$ will be denoted by $\mathcal{F}\mathfrak{Nano}ZO(V)$ (resp. $\mathcal{F}\mathfrak{Nano}ZC(V)$).

Definition 2.2 A $\mathcal{F}subs$ S in V is said to be a fuzzy nano

(i) Z closure of S (briefly, $\mathcal{F}\mathfrak{Nano}Zcl(S)$) is described as $\mathcal{F}\mathfrak{Nano}Zcl(S) = \bigwedge\{C : S \leqslant C$ & C is a $\mathcal{F}\mathfrak{Nano}Zc$ set$\}$.
(ii) Z interior of S (briefly, $\mathcal{F}\mathfrak{Nano}Zint(S)$) is described as $\mathcal{F}\mathfrak{Nano}Zint(S) = \bigvee\{C : C \leqslant S$ & C is a $\mathcal{F}\mathfrak{Nano}Zo$ set$\}$.

Remark 1 *Let L be a $\mathcal{F}$subs of a $\mathcal{F}\mathfrak{Nano}ts$ $(V, \tau_{\mathcal{F}}(F))$. Then*

$$(\mathcal{F}\mathfrak{Nano}Zcl(L))^c = \mathcal{F}\mathfrak{Nano}Zint(L^c), (\mathcal{F}\mathfrak{Nano}Zint(L))^c = \mathcal{F}\mathfrak{Nano}Zcl(L^c).$$

Theorem 2.3 *The following statements are true. Every*

(i) $\mathcal{F}\mathfrak{Nano}\delta o$ *(resp.* $\mathcal{F}\mathfrak{Nano}\delta c$*) set is* $\mathcal{F}\mathfrak{Nano}o$ *(resp.* $\mathcal{F}\mathfrak{Nano}c$*) set.*
(ii) $\mathcal{F}\mathfrak{Nano}o$ *(resp.* $\mathcal{F}\mathfrak{Nano}c$*) set is* $\mathcal{F}\mathfrak{Nano}\mathcal{P}o$ *(resp.* $\mathcal{F}\mathfrak{Nano}\mathcal{P}c$*) set.*
(iii) $\mathcal{F}\mathfrak{Nano}\delta o$ *(resp.* $\mathcal{F}\mathfrak{Nano}\delta c$*) set is* $\mathcal{F}\mathfrak{Nano}\delta\mathcal{S}o$ *(resp.* $\mathcal{F}\mathfrak{Nano}\delta\mathcal{S}c$*) set.*
(iv) $\mathcal{F}\mathfrak{Nano}\delta\mathcal{S}o$ *(resp.* $\mathcal{F}\mathfrak{Nano}\delta\mathcal{S}c$*) set is* $\mathcal{F}\mathfrak{Nano}Zo$ *(resp.* $\mathcal{F}\mathfrak{Nano}Zc$*) set.*
(v) $\mathcal{F}\mathfrak{Nano}\mathcal{P}o$ *(resp.* $\mathcal{F}\mathfrak{Nano}\mathcal{P}c$*) set is* $\mathcal{F}\mathfrak{Nano}Zo$ *(resp.* $\mathcal{F}\mathfrak{Nano}Zc$*) set.*

But not conversely.

Proof. (i) Let S is a $\mathcal{F}\mathfrak{Nano}\delta o$, then $S = \mathcal{F}\mathfrak{Nano}\delta int(S) \leqslant \mathcal{F}\mathfrak{Nano}int(S)$. Therefore S is a $\mathcal{F}\mathfrak{Nano}o$.

(ii) Let S is a $\mathcal{F}\mathfrak{Nano}o$, then $S = \mathcal{F}\mathfrak{Nano}int(S) \leqslant \mathcal{F}\mathfrak{Nano}int(\mathcal{F}\mathfrak{Nano}cl(S))$. Therefore S is a $\mathcal{F}\mathfrak{Nano}\mathcal{P}o$.

(iii) Let S is a $\mathcal{F}\mathfrak{Nano}o$, then $S = \mathcal{F}\mathfrak{Nano}int(S) \leqslant \mathcal{F}\mathfrak{Nano}cl(\mathcal{F}\mathfrak{Nano}\delta int(S))$. Therefore S is a $\mathcal{F}\mathfrak{Nano}\delta\mathcal{S}o$.

(iv) S is a $\mathcal{F}\mathfrak{Nano}\delta\mathcal{S}os$, then $S \leqslant \mathcal{F}\mathfrak{Nano}cl(\mathcal{F}\mathfrak{Nano}\delta int(S))$ and so $S \leqslant \mathcal{F}\mathfrak{Nano}cl(\mathcal{F}\mathfrak{Nano}\delta int(S)) \leqslant \mathcal{F}\mathfrak{Nano}cl(\mathcal{F}\mathfrak{Nano}\delta int(S)) \vee \mathcal{F}\mathfrak{Nano}int(\mathcal{F}\mathfrak{Nano}cl(S))$. Therefore S is a $\mathcal{F}\mathfrak{Nano}Zos$.

(v) S is a $\mathcal{F}\mathfrak{Nano}\mathcal{P}os$, then $S \leqslant \mathcal{F}\mathfrak{Nano}int(\mathcal{F}\mathfrak{Nano}cl(S))$ and so $S \leqslant \mathcal{F}\mathfrak{Nano}int(\mathcal{F}\mathfrak{Nano}cl(S)) \leqslant \mathcal{F}\mathfrak{Nano}int(\mathcal{F}\mathfrak{Nano}cl(S)) \vee \mathcal{F}\mathfrak{Nano}cl(\mathcal{F}\mathfrak{Nano}\delta int(S))$. Therefore S is a $\mathcal{F}\mathfrak{Nano}Zos$. ■

Remark 2 *The following diagram holds for any set in $\mathcal{F}\mathfrak{Nano}ts$, according to Definition 2.1 and Theorem 2.3.*

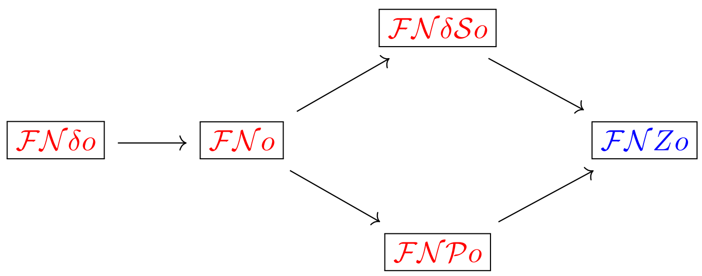

Figure 1. $\mathcal{FN}Zos$'s in $\mathcal{FN}ts$.

Proposition 2.4 Let $(V, \tau_{\mathcal{F}}(F))$ be a $\mathcal{F}\mathfrak{Nano}ts$ with respect to F where F is a fuzzy subset of V. Then for any fuzzy subset M of V, the following hold.

(i) $\mathcal{F}\mathfrak{Nano}\delta int(M) \leqslant \mathcal{F}\mathfrak{Nano}\delta\mathcal{S}int(M) \leqslant \mathcal{F}\mathfrak{Nano}Zint(M)$.
(ii) $\mathcal{F}\mathfrak{Nano}\delta int(M) \leqslant \mathcal{F}\mathfrak{Nano}int(M) \leqslant \mathcal{F}\mathfrak{Nano}\mathcal{P}int(M) \leqslant \mathcal{F}\mathfrak{Nano}Zint(M)$.
(iii) $\mathcal{F}\mathfrak{Nano}\delta cl(M) \geqslant \mathcal{F}\mathfrak{Nano}\delta\mathcal{S}cl(M) \geqslant \mathcal{F}\mathfrak{Nano}Zcl(M)$.
(iv) $\mathcal{F}\mathfrak{Nano}\delta cl(M) \geqslant \mathcal{F}\mathfrak{Nano}cl(M) \geqslant \mathcal{F}\mathfrak{Nano}\mathcal{P}cl(M) \geqslant \mathcal{F}\mathfrak{Nano}Zcl(M)$.

Example **2.5** Assume $U = \{s_1, s_2, s_3, s_4\}$ and $U/R = \{\{s_1, s_4\}, \{s_2\}, \{s_3\}\}$. Let $S =$

$\{\langle\frac{s_1}{0.2}\rangle, \langle\frac{s_2}{0.3}\rangle, \langle\frac{s_3}{0.4}\rangle, \langle\frac{s_4}{0.1}\rangle\}$ be a $\mathcal{F}subs$ of U.

$$\underline{\mathcal{F}\mathfrak{Nano}}(S) = \left\{\left\langle\frac{s_1, s_4}{0.1}\right\rangle, \left\langle\frac{s_2}{0.3}\right\rangle, \left\langle\frac{s_3}{0.4}\right\rangle\right\},$$
$$\overline{\mathcal{F}\mathfrak{Nano}}(S) = \left\{\left\langle\frac{s_1, s_4}{0.2}\right\rangle, \left\langle\frac{s_2}{0.3}\right\rangle, \left\langle\frac{s_3}{0.4}\right\rangle\right\},$$
$$B_{\mathcal{F}\mathfrak{Nano}}(S) = \left\{\left\langle\frac{s_1, s_4}{0.2}\right\rangle, \left\langle\frac{s_2}{0.3}\right\rangle, \left\langle\frac{s_3}{0.4}\right\rangle\right\}.$$

Thus $\tau_{\mathcal{F}}(S) = \{0_{\mathcal{F}}, 1_{\mathcal{F}}, \underline{\mathcal{F}\mathfrak{Nano}}(S), \overline{\mathcal{F}\mathfrak{Nano}}(S) = B_{\mathcal{F}\mathfrak{Nano}}(S)\}$.
Then

(i) $\{\langle\frac{s_1,s_4}{0.2}\rangle, \langle\frac{s_2}{0.3}\rangle, \langle\frac{s_3}{0.6}\rangle\}$ is a $\mathcal{F}\mathfrak{Nano}Zo$ set but not $\mathcal{F}\mathfrak{Nano}\mathcal{P}o$ set.
(ii) $\{\langle\frac{s_1,s_4}{0.3}\rangle, \langle\frac{s_2}{0.4}\rangle, \langle\frac{s_3}{0.7}\rangle\}$ is a $\mathcal{F}\mathfrak{Nano}Zo$ set but not $\mathcal{F}\mathfrak{Nano}\delta\mathcal{S}o$ set.

Lemma 2.6 Let $(V, \tau_{\mathcal{F}}(F))$ be a $\mathcal{F}\mathfrak{Nano}ts$. Then the union (resp. intersection) of arbitrary $\mathcal{F}\mathfrak{Nano}Zo$ (resp. $\mathcal{F}\mathfrak{Nano}Zc$) sets is $\mathcal{F}\mathfrak{Nano}Zo$ (resp. $\mathcal{F}\mathfrak{Nano}Zc$).

Proof. Let $\{S_i, i \in I\}$ be a family of $\mathcal{F}\mathfrak{Nano}Zo$ sets. Then $S_i \leqslant \mathcal{F}\mathfrak{Nano}cl(\mathcal{F}\mathfrak{Nano}\,\delta int(S_i)) \vee \mathcal{F}\mathfrak{Nano}int(\mathcal{F}\mathfrak{Nano}cl(S_i))$ and hence $\bigvee_i S_i \leqslant \bigvee_i(\mathcal{F}\mathfrak{Nano}cl(\mathcal{F}\mathfrak{Nano}\delta\, int(S_i)) \vee \mathcal{F}\mathfrak{Nano}int(\mathcal{F}\mathfrak{Nano}cl(S_i))) \leqslant \mathcal{F}\mathfrak{Nano}cl(\mathcal{F}\mathfrak{Nano}\delta int(\bigvee_i S_i)) \vee \mathcal{F}\mathfrak{Nano}int\,(\mathcal{F}\mathfrak{Nano}cl(\bigvee_i S_i))$, $\forall\, i \in I$. Thus $\bigvee_i S_i$ is $\mathcal{F}\mathfrak{Nano}Zo$.

The other is comparable. ■

Remark 3 *The intersection of any two $\mathcal{F}\mathfrak{Nano}Zo$ sets is not $\mathcal{F}\mathfrak{Nano}Zo$, as shown in the following.*

***Example* 2.7** In Example 2.5, let $A = \{\langle\frac{s_1,s_4}{0.33}\rangle, \langle\frac{s_2}{0.53}\rangle, \langle\frac{s_3}{0.53}\rangle\}$ and $B = \{\langle\frac{s_1,s_4}{0.11}\rangle, \langle\frac{s_2}{0.22}\rangle, \langle\frac{s_3}{0.77}\rangle\}$ are $\mathcal{F}\mathfrak{Nano}Zo$ sets but $A \wedge B = \{\langle\frac{s_1,s_4}{0.11}\rangle, \langle\frac{s_2}{0.22}\rangle, \langle\frac{s_3}{0.55}\rangle\}$ is not $\mathcal{F}\mathfrak{Nano}Zo$ set.

Theorem 2.8 Let S be a $\mathcal{F}\mathfrak{Nano}Zc$ set in $(V, \tau_{\mathcal{F}}(F))$. Then $\mathcal{F}\mathfrak{Nano}Zcl(S) - S$ contains no non-empty spaces $\mathcal{F}\mathfrak{Nano}c$ set in $(V, \tau_{\mathcal{F}}(F))$.

Proof. Let S be a $\mathcal{F}\mathfrak{Nano}Zc$ set in $(V, \tau_{\mathcal{F}}(F))$ and M be a $\mathcal{F}\mathfrak{Nano}c$ subset of $\mathcal{F}\mathfrak{Nano}Zcl(S) - S$. i.e., $M \leqslant \mathcal{F}\mathfrak{Nano}Zcl(S) - S$ implies $M \leqslant \mathcal{F}\mathfrak{Nano}Zcl(S) \wedge (1_{\mathcal{F}} - S)$. That is $M \leqslant \mathcal{F}\mathfrak{Nano}Zcl(S)$ and $M \leqslant (1_{\mathcal{F}} - S)$ which implies $S \leqslant (1_{\mathcal{F}} - M)$ where $1_{\mathcal{F}} - M$ is a $\mathcal{F}\mathfrak{Nano}o$ set. Since S is $\mathcal{F}\mathfrak{Nano}Zc$, $\mathcal{F}\mathfrak{Nano}Zcl(S) \leqslant (1_{\mathcal{F}} - M)$. That is $M \leqslant (1_{\mathcal{F}} - \mathcal{F}\mathfrak{Nano}Zcl(S))$. Thus $M \leqslant \mathcal{F}\mathfrak{Nano}Zcl(S) \wedge (1_{\mathcal{F}} - \mathcal{F}\mathfrak{Nano}Zcl(S)) = \phi$. Hence $M = \phi$. Therefore $\mathcal{F}\mathfrak{Nano}Zcl(S) - S$ contains no non-empty spaces $\mathcal{F}\mathfrak{Nano}c$ set in $(V, \tau_{\mathcal{F}}(F))$. ■

Theorem 2.9 If S is a $\mathcal{F}\mathfrak{Nano}Zc$ set in $(V, \tau_{\mathcal{F}}(F))$, then S is $\mathcal{F}\mathfrak{Nano}Zc$ if and only if $\mathcal{F}\mathfrak{Nano}Zcl(S) - S$ is a $\mathcal{F}\mathfrak{Nano}c$ set in $(V, \tau_{\mathcal{F}}(F))$.

Proof. Let S be a set of $\mathcal{F}\mathfrak{Nano}Zc$. Assume that S is $\mathcal{F}\mathfrak{Nano}Zc$ we have $\mathcal{F}\mathfrak{Nano}Zcl(S) = S$, $\mathcal{F}\mathfrak{Nano}Zcl(S) - S = \phi$ which is $\mathcal{F}\mathfrak{Nano}c$.

Conversely, assume that $\mathcal{F}\mathfrak{Nano}Zcl(S) - S$ is a $\mathcal{F}\mathfrak{Nano}c$ and S is a $\mathcal{F}\mathfrak{Nano}Zc$ set in V. Now $\mathcal{F}\mathfrak{Nano}Zcl(S) - S$ is a subset of itself $\mathcal{F}\mathfrak{Nano}c$. As a result of the Theorem 2.8, $\mathcal{F}\mathfrak{Nano}Zcl(S) - S = \phi$. That is $\mathcal{F}\mathfrak{Nano}Zcl(S) = S$, S must be $\mathcal{F}\mathfrak{Nano}Zc$. ■

Theorem 2.10 If S is $\mathcal{F}\mathfrak{Nano}Zc$ set in $(V, \tau_{\mathcal{F}}(F))$ and $S \leqslant O \leqslant \mathcal{F}\mathfrak{Nano}Zcl(S)$, then O is also $\mathcal{F}\mathfrak{Nano}Zc$ in $(V, \tau_{\mathcal{F}}(F))$.

Proof. Let S be a $\mathcal{F}\mathfrak{Nano}Zc$ set in V, with $S \leqslant O \leqslant \mathcal{F}\mathfrak{Nano}Zcl(S)$. Let $O \leqslant M$

be $\mathcal{F}\mathfrak{Nano}o$ set in V, and M be $\mathcal{F}\mathfrak{Nano}o$. Since $S \leqslant O$, $S < M$ follows, and S is $\mathcal{F}\mathfrak{Nano}Zc$, $\mathcal{F}\mathfrak{Nano}Zcl(S) \leqslant M$ follows. According to the hypothesis $O \leqslant \mathcal{F}\mathfrak{Nano}Zcl(S)$, $\mathcal{F}\mathfrak{Nano}Zcl(O) \leqslant \mathcal{F}\mathfrak{Nano}Zcl(\mathcal{F}\mathfrak{Nano}Zcl(S)) = \mathcal{F}\mathfrak{Nano}Zcl(S) \leqslant M$, which implies $\mathcal{F}\mathfrak{Nano}Zcl(O) \leqslant M$. Therefore O is $\mathcal{F}\mathfrak{Nano}Zc$ in V. ∎

Theorem 2.11 If a subset S of V is $\mathcal{F}\mathfrak{Nano}Zc$ set, then

$$\mathcal{F}\mathfrak{Nano}cl(\{x_r\}) \wedge S \neq \phi \forall x_r \in \mathcal{F}\mathfrak{Nano}Zcl(S).$$

Proof. Assume that S is a $\mathcal{F}\mathfrak{Nano}Zc$ set and that $x_r \in \mathcal{F}\mathfrak{Nano}Zcl(S)$. If possible, $\mathcal{F}\mathfrak{Nano}cl(\{x_r\}) \wedge S = \phi$. Then $S \leqslant 1_{\mathcal{F}} - \mathcal{F}\mathfrak{Nano}cl(\{x_r\})$ and $1_{\mathcal{F}} - \mathcal{F}\mathfrak{Nano}cl(\{x_r\})$ is a $\mathcal{F}\mathfrak{Nano}o$ set containing S. Since S is $\mathcal{F}\mathfrak{Nano}Zc$ set, implies $\mathcal{F}\mathfrak{Nano}Zcl(S) \leqslant 1_{\mathcal{F}} - \mathcal{F}\mathfrak{Nano}cl(\{x_r\})$ which is a contradiction to $x_r \in \mathcal{F}\mathfrak{Nano}Zcl(S)$. Therefore, $\mathcal{F}\mathfrak{Nano}cl(\{x_r\}) \wedge S \neq \phi$. ∎

Theorem 2.12 If $\mathcal{F}\mathfrak{Nano}ZO(V) = \mathcal{F}\mathfrak{Nano}ZC(V)$, then $\mathcal{F}\mathfrak{Nano}ZC(V) = P(V)$ where $P(V)$ is the power set of V.

Definition 2.13 The intersection of all $\mathcal{F}\mathfrak{Nano}o$ subset of V containing S is called the fuzzy nano kernel of S (briefly, $\mathcal{F}\mathfrak{Nano}ker(S)$), this means

$$\mathcal{F}\mathfrak{Nano}ker(S) = \wedge\{G \in \mathcal{F}\mathfrak{Nano}O(V) : S \leqslant G\}.$$

Theorem 2.14 A subset S is a $\mathcal{F}\mathfrak{Nano}Zc$ set iff $\mathcal{F}\mathfrak{Nano}Zcl(S) \leqslant \mathcal{F}\mathfrak{Nano}ker(S)$.

Proof. Suppose S is $\mathcal{F}\mathfrak{Nano}Zc$ set, then $\mathcal{F}\mathfrak{Nano}Zcl(S) \leqslant M$ whenever $S \leqslant M$ and M is $\mathcal{F}\mathfrak{Nano}o$. Let $x_r \in \mathcal{F}\mathfrak{Nano}Zcl(S)$. If $x_r \notin \mathcal{F}\mathfrak{Nano}ker(S)$, then $\exists$ a $\mathcal{F}\mathfrak{Nano}o$ set M containing $S \ni x_r \notin M$. Since M is a $\mathcal{F}\mathfrak{Nano}o$ set containing S, implies $x_r \notin \mathcal{F}\mathfrak{Nano}Zcl(S)$, which is a contradiction. Thus, $x_r \in \mathcal{F}\mathfrak{Nano}ker(S)$. Conversely, let $\mathcal{F}\mathfrak{Nano}Zcl(S) \leqslant \mathcal{F}\mathfrak{Nano}ker(S)$. If M is a $\mathcal{F}\mathfrak{Nano}o$ set containing S, then $\mathcal{F}\mathfrak{Nano}ker(S) \leqslant M$, which implies $\mathcal{F}\mathfrak{Nano}Zcl(S) \leqslant M$. Therefore, S is a $\mathcal{F}\mathfrak{Nano}Zc$ set. ∎

Theorem 2.15 If $S \leqslant W \leqslant V$ and S is a $\mathcal{F}\mathfrak{Nano}Zc$ set in V, then S is a $\mathcal{F}\mathfrak{Nano}Zc$ in W.

Lemma 2.16 Let S, L be two $\mathcal{F}subs$'s of $(V, \tau_{\mathcal{F}}(F))$. Then

(i) $1_{\mathcal{F}} - \mathcal{F}\mathfrak{Nano}\delta int(S) = \mathcal{F}\mathfrak{Nano}\delta cl(1_{\mathcal{F}} - S)$ and $\mathcal{F}\mathfrak{Nano}\delta int(1_{\mathcal{F}} - S) = 1_{\mathcal{F}} - \mathcal{F}\mathfrak{Nano}\delta cl(S)$,
(ii) $\mathcal{F}\mathfrak{Nano}cl(S) \leqslant \mathcal{F}\mathfrak{Nano}\delta cl(S)$ (resp. $\mathcal{F}\mathfrak{Nano}\delta int(S) \leqslant \mathcal{F}\mathfrak{Nano}int(S)$), for any subset S of V,
(iii) $\mathcal{F}\mathfrak{Nano}\delta cl(S \vee L) = \mathcal{F}\mathfrak{Nano}\delta cl(S) \vee \mathcal{F}\mathfrak{Nano}\delta cl(L)$, $\mathcal{F}\mathfrak{Nano}\delta int(S \wedge L) = \mathcal{F}\mathfrak{Nano}\delta int(S) \wedge \mathcal{F}\mathfrak{Nano}\delta int(L)$.

Proposition 2.17 Let S be a $\mathcal{F}subs$ in a $\mathcal{F}\mathfrak{Nano}ts$ $(V, \tau_{\mathcal{F}}(F))$. Then:

(i) $\mathcal{F}\mathfrak{Nano}\mathcal{P}cl(S) = S \vee \mathcal{F}\mathfrak{Nano}cl(\mathcal{F}\mathfrak{Nano}int(S))$, $\mathcal{F}\mathfrak{Nano}\mathcal{P}int(S) = S \wedge \mathcal{F}\mathfrak{Nano}int(\mathcal{F}\mathfrak{Nano}cl(S))$,
(ii) $\mathcal{F}\mathfrak{Nano}\delta Scl(1_{\mathcal{F}} - S) = 1_{\mathcal{F}} - \mathcal{F}\mathfrak{Nano}\delta Sint(S)$, $\mathcal{F}\mathfrak{Nano}\delta Scl(S \vee L) \leqslant \mathcal{F}\mathfrak{Nano}\delta Scl(S) \vee \mathcal{F}\mathfrak{Nano}\delta Scl(L)$.

Lemma 2.18 The following hold for a $\mathcal{F}subs$ S in a $\mathcal{F}\mathfrak{Nano}ts$ $(V, \tau_{\mathcal{F}}(F))$.

(i) $\mathcal{F}\mathfrak{Nano}\delta\mathcal{P}cl(S) = S \wedge \mathcal{F}\mathfrak{Nano}cl(\mathcal{F}\mathfrak{Nano}\delta int(S))$ and $\mathcal{F}\mathfrak{Nano}\delta\mathcal{P}int(S) = S \vee \mathcal{F}\mathfrak{Nano}int(\mathcal{F}\mathfrak{Nano}\delta cl(S))$,

(ii) $\mathcal{F}\mathfrak{Nano}\delta\mathcal{S}int(S) = S \wedge \mathcal{F}\mathfrak{Nano}cl(\mathcal{F}\mathfrak{Nano}\delta int(S))$ and $\mathcal{F}\mathfrak{Nano}\delta\mathcal{S}cl(S) = S \vee \mathcal{F}\mathfrak{Nano}int(\mathcal{F}\mathfrak{Nano}\delta cl(S))$.

Lemma 2.19 The following hold for a $\mathcal{F}subs$ S in a $\mathcal{F}\mathfrak{Nano}ts$ $(V, \tau_{\mathcal{F}}(F))$.

$$\mathcal{F}\mathfrak{Nano}cl(\mathcal{F}\mathfrak{Nano}\delta int(S)) = \mathcal{F}\mathfrak{Nano}\delta cl(\mathcal{F}\mathfrak{Nano}\delta int(S))$$

and

$$\mathcal{F}\mathfrak{Nano}int(\mathcal{F}\mathfrak{Nano}\delta cl(S)) = \mathcal{F}\mathfrak{Nano}\delta int(\mathcal{F}\mathfrak{Nano}\delta cl(S)).$$

Theorem 2.20

(i) If $S \in \mathcal{F}\mathfrak{Nano}\delta O(V)$ and $L \in \mathcal{F}\mathfrak{Nano}ZO(V)$, then $S \wedge L$ is $\mathcal{F}\mathfrak{Nano}Zo$,
(ii) If $S \in \mathcal{F}\mathfrak{Nano}aO(V)$ and $L \in \mathcal{F}\mathfrak{Nano}ZO(V)$, then $S \wedge L \in \mathcal{F}\mathfrak{Nano}ZO(V)$.

Proof. (i) Suppose that $S \in \mathcal{F}\mathfrak{Nano}\delta O(V)$. Then $S = \mathcal{F}\mathfrak{Nano}\delta int(S)$. Since $L \in \mathcal{F}\mathfrak{Nano}ZO(V)$, then

$$L \leqslant \mathcal{F}\mathfrak{Nano}cl(\mathcal{F}\mathfrak{Nano}\delta int(L)) \vee \mathcal{F}\mathfrak{Nano}int(\mathcal{F}\mathfrak{Nano}cl(L))$$

and hence

$$\begin{aligned} S \wedge L &\leqslant \mathcal{F}\mathfrak{Nano}\delta int(S) \wedge (\mathcal{F}\mathfrak{Nano}cl(\mathcal{F}\mathfrak{Nano}\delta int(L)) \vee \mathcal{F}\mathfrak{Nano}int(\mathcal{F}\mathfrak{Nano}cl(L))) \\ &= (\mathcal{F}\mathfrak{Nano}\delta int(S) \wedge \mathcal{F}\mathfrak{Nano}cl(\mathcal{F}\mathfrak{Nano}\delta int(L))) \vee (\mathcal{F}\mathfrak{Nano}\delta int(S) \wedge \mathcal{F}\mathfrak{Na}\text{-} \\ &\quad \mathfrak{no}int(\mathcal{F}\mathfrak{Nano}cl(L))) \\ &\leqslant \mathcal{F}\mathfrak{Nano}cl(\mathcal{F}\mathfrak{Nano}\delta int(S) \wedge (\mathcal{F}\mathfrak{Nano}\delta int(L))) \vee \mathcal{F}\mathfrak{Nano}int(\mathcal{F}\mathfrak{Nano}int(S) \\ &\quad \wedge \mathcal{F}\mathfrak{Nano}cl(L)) \\ &\leqslant \mathcal{F}\mathfrak{Nano}cl(\mathcal{F}\mathfrak{Nano}\delta int(S \wedge L)) \vee \mathcal{F}\mathfrak{Nano}int(\mathcal{F}\mathfrak{Nano}cl(S \wedge L)). \end{aligned}$$

Thus $S \wedge L \leqslant \mathcal{F}\mathfrak{Nano}cl(\mathcal{F}\mathfrak{Nano}\delta int(S \wedge L)) \vee \mathcal{F}\mathfrak{Nano}int(\mathcal{F}\mathfrak{Nano}cl(S \wedge L))$. Therefore, $S \wedge L$ is $\mathcal{F}\mathfrak{Nano}Zo$.

(ii) Since

$$\begin{aligned} S \wedge L &\leqslant \mathcal{F}\mathfrak{Nano}int(\mathcal{F}\mathfrak{Nano}cl(\mathcal{F}\mathfrak{Nano}\delta int(S))) \wedge (\mathcal{F}\mathfrak{Nano}cl(\mathcal{F}\mathfrak{Nano}\delta int(L)) \\ &\quad \vee \mathcal{F}\mathfrak{Nano}int(\mathcal{F}\mathfrak{Nano}cl(L))) \\ &= (\mathcal{F}\mathfrak{Nano}int(\mathcal{F}\mathfrak{Nano}cl(\mathcal{F}\mathfrak{Nano}\delta int(S))) \wedge \mathcal{F}\mathfrak{Nano}cl(\mathcal{F}\mathfrak{Nano}\delta int(L))) \\ &\quad \vee (\mathcal{F}\mathfrak{Nano}int(\mathcal{F}\mathfrak{Nano}cl(\mathcal{F}\mathfrak{Nano}\delta int(S))) \wedge \mathcal{F}\mathfrak{Nano}int(\mathcal{F}\mathfrak{Nano}cl(L))) \\ &\leqslant \mathcal{F}\mathfrak{Nano}cl(\mathcal{F}\mathfrak{Nano}cl(\mathcal{F}\mathfrak{Nano}\delta int(S)) \wedge \mathcal{F}\mathfrak{Nano}\delta int(L)) \vee \mathcal{F}\mathfrak{Nano} \\ &\quad int(\mathcal{F}\mathfrak{Nano}cl(\mathcal{F}\mathfrak{Nano}\delta int(S)) \wedge \mathcal{F}\mathfrak{Nano}int(\mathcal{F}\mathfrak{Nano}cl(L))) \\ &\leqslant \mathcal{F}\mathfrak{Nano}cl(\mathcal{F}\mathfrak{Nano}cl(\mathcal{F}\mathfrak{Nano}\delta int(S) \wedge \mathcal{F}\mathfrak{Nano}\delta int(L))) \vee \mathcal{F}\mathfrak{Nano} \\ &\quad int(\mathcal{F}\mathfrak{Nano}cl(\mathcal{F}\mathfrak{Nano}\delta int(S)) \wedge \mathcal{F}\mathfrak{Nano}int(\mathcal{F}\mathfrak{Nano}cl(L))) \end{aligned}$$

and hence

$$\begin{aligned} S \wedge L &\leqslant (S \wedge \mathcal{F}\mathfrak{Nano}cl(\mathcal{F}\mathfrak{Nano}\delta int(S) \wedge \mathcal{F}\mathfrak{Nano}\delta int(L))) \vee (S \wedge \mathcal{F}\mathfrak{Nano}int \\ &\quad (\mathcal{F}\mathfrak{Nano}cl(\mathcal{F}\mathfrak{Nano}\delta int(S)) \wedge \mathcal{F}\mathfrak{Nano}int(\mathcal{F}\mathfrak{Nano}cl(L)))) \\ &\leqslant \mathcal{F}\mathfrak{Nano}cl(\mathcal{F}\mathfrak{Nano}\delta int(S) \wedge \mathcal{F}\mathfrak{Nano}\delta int(L)) \vee \mathcal{F}\mathfrak{Nano}int(\mathcal{F}\mathfrak{Nano}cl \\ &\quad (\mathcal{F}\mathfrak{Nano}\delta int(S) \wedge \mathcal{F}\mathfrak{Nano}int(\mathcal{F}\mathfrak{Nano}cl(L))))\mathcal{F}\mathfrak{Nano}int(S) \wedge L)) \\ &\leqslant \mathcal{F}\mathfrak{Nano}cl(\mathfrak{Nano}\delta int) \wedge \mathcal{F}\mathfrak{Nano}\delta int(L) \vee \mathcal{F}\mathfrak{Nano}int(S)(\mathcal{F}\mathfrak{Nano}cl(\mathcal{F} \\ &\quad \mathfrak{Nano}\delta int(S) \wedge \mathcal{F}\mathfrak{Nano}cl(L))) \\ &\leqslant \mathcal{F}\mathfrak{Nano}cl(\mathfrak{Nano}\delta int \wedge \mathcal{F}\mathfrak{Nano}\delta int(L)) \vee \mathcal{F}\mathfrak{Nano}int(\mathcal{F}\mathfrak{Nano}cl(\mathcal{F}\mathfrak{Nano} \\ &\quad cl(\mathcal{F}\mathfrak{Nano}\delta int(S) \wedge L))). \end{aligned}$$

Since $\mathfrak{Nano}\delta int(S) \wedge \mathcal{F}\mathfrak{Nano}\delta int(L) \leqslant \mathcal{F}\mathfrak{Nano}\delta int(S) \leqslant S$ which is $\mathcal{F}\mathfrak{Nano}\delta o$ in S, then

$$\begin{aligned} S \wedge L &\leqslant \mathcal{F}\mathfrak{Nano}cl\mathfrak{Nano}\delta int(\mathcal{F}\mathfrak{Nano}\delta int(S) \wedge \mathcal{F}\mathfrak{Nano}\delta int(L)) \vee \mathcal{F}\mathfrak{Nano}int \\ &\quad (S \vee \mathcal{F}\mathfrak{Nano}cl(\mathcal{F}\mathfrak{Nano}\delta int(S) \wedge L)) \\ &\leqslant \mathcal{F}\mathfrak{Nano}cl\mathfrak{Nano}\delta int(S \wedge L) \vee \mathcal{F}\mathfrak{Nano}int\mathcal{F}\mathfrak{Nano}cl(\mathcal{F}\mathfrak{Nano}\delta int(S) \wedge L) \\ &\leqslant \mathcal{F}\mathfrak{Nano}cl\mathfrak{Nano}\delta int(S \wedge L) \vee \mathcal{F}\mathfrak{Nano}int\mathcal{F}\mathfrak{Nano}cl(S \wedge L). \end{aligned}$$

Therefore $S \wedge L \in \mathcal{F}\mathfrak{Nano}ZO(V)$. ■

Proposition 2.21 The closure of a $\mathcal{F}\mathfrak{Nano}Zo$ set of A is $\mathcal{F}\mathfrak{Nano}\delta\mathcal{S}o$.

Proof. Let $S \in \mathcal{F}\mathfrak{Nano}ZO(A)$. Then $\mathcal{F}\mathfrak{Nano}cl(S) \leqslant \mathcal{F}\mathfrak{Nano}cl(\mathcal{F}\mathfrak{Nano}cl(\mathcal{F}\mathfrak{Nano}\delta int(S)) \vee \mathcal{F}\mathfrak{Nano}int(\mathcal{F}\mathfrak{Nano}cl(S))) \leqslant \mathcal{F}\mathfrak{Nano}cl(\mathcal{F}\mathfrak{Nano}\delta int(S)) \vee \mathcal{F}\mathfrak{Nano}cl(\mathcal{F}\mathfrak{Nano}int(\mathcal{F}\mathfrak{Nano}cl(S))) = \mathcal{F}\mathfrak{Nano}cl(\mathcal{F}\mathfrak{Nano}\delta int(S))$.

Therefore, $\mathcal{F}\mathfrak{Nano}cl(S)$ is $\mathcal{F}\mathfrak{Nano}\delta\mathcal{S}o$. ■

Proposition 2.22 Let S be a $\mathcal{F}\mathfrak{Nano}Zo$ subset of a $\mathcal{F}\mathfrak{Nano}ts$ $(V, \tau_{\mathcal{F}}(F))$ and $\mathcal{F}\mathfrak{Nano}\delta int(S) = \phi$. Then S is $\mathcal{F}\mathfrak{Nano}\mathcal{P}o$.

Theorem 2.23 Let S and L be two subsets of a $\mathcal{F}\mathfrak{Nano}ts$ $(V, \tau\mathcal{F}(F))$. The following holds true:

(i) $\mathcal{F}\mathfrak{Nano}Zcl(1_{\mathcal{F}} - S) = 1_{\mathcal{F}} - \mathcal{F}\mathfrak{Nano}Zint(S)$,
(ii) $\mathcal{F}\mathfrak{Nano}Zint(1_{\mathcal{F}} - S) = 1_{\mathcal{F}} - \mathcal{F}\mathfrak{Nano}Zcl(S)$,
(iii) If $S \leqslant L$, then $\mathcal{F}\mathfrak{Nano}Zcl(S) \leqslant \mathcal{F}\mathfrak{Nano}Zcl(L)$ and $\mathcal{F}\mathfrak{Nano}Zint(S) \leqslant \mathcal{F}\mathfrak{Nano}Zint(L)$,
(iv) $l \in \mathcal{F}\mathfrak{Nano}Zcl(S)$ iff $\forall$ a $\mathcal{F}\mathfrak{Nano}Zo$ set A contains l, $A \wedge S \neq \phi$,
(v) $l \in \mathcal{F}\mathfrak{Nano}Zint(S)$ iff $\exists$ a $\mathcal{F}\mathfrak{Nano}Zo$ set $H \ni l \in H \leqslant S$,
(vi) $\mathcal{F}\mathfrak{Nano}Zcl(\mathcal{F}\mathfrak{Nano}Zcl(S)) = \mathcal{F}\mathfrak{Nano}Zcl(S)$ and $\mathcal{F}\mathfrak{Nano}Zint(\mathcal{F}\mathfrak{Nano}Zint\ (S)) = \mathcal{F}\mathfrak{Nano}Zint(S)$,
(vii) $\mathcal{F}\mathfrak{Nano}Zcl(S) \vee \mathcal{F}\mathfrak{Nano}Zcl(L) \leqslant \mathcal{F}\mathfrak{Nano}Zcl(S \vee L)$ and $\mathcal{F}\mathfrak{Nano}Zint(S) \vee \mathcal{F}\mathfrak{Nano}Zint(L) \leqslant \mathcal{F}\mathfrak{Nano}Zint(S \vee L)$,
(viii) $\mathcal{F}\mathfrak{Nano}Zint(S \wedge L) \leqslant \mathcal{F}\mathfrak{Nano}Zint(S) \wedge \mathcal{F}\mathfrak{Nano}Zint(L)$ and $\mathcal{F}\mathfrak{Nano}Zcl(S \wedge L) \leqslant \mathcal{F}\mathfrak{Nano}Zcl(S) \wedge \mathcal{F}\mathfrak{Nano}Zcl(L)$.

Proof. Definition 2.2 has shown this. ■

Remark 4 *As illustrated in the following example, the inclusion relation in (vii) and (viii) of the preceding theorem cannot be replaced by equality.*

***Example* 2.24** In Example 2.5, the sets

(i) $A = \{\langle\frac{s_1,s_4}{0.1}\rangle, \langle\frac{s_2}{0.5}\rangle, \langle\frac{s_3}{0.4}\rangle\}$ and $B = \{\langle\frac{s_1,s_4}{0.2}\rangle, \langle\frac{s_2}{0.4}\rangle, \langle\frac{s_3}{0.6}\rangle\}$, then
$A \vee B = \{\langle\frac{s_1,s_4}{0.2}\rangle, \langle\frac{s_2}{0.5}\rangle, \langle\frac{s_3}{0.6}\rangle\}$.
$\mathcal{F}\mathfrak{Nano}Zint(A) = \{\langle\frac{s_1,s_4}{0.1}\rangle, \langle\frac{s_2}{0.3}\rangle, \langle\frac{s_3}{0.4}\rangle\}$,
$\mathcal{F}\mathfrak{Nano}Zint(B) = \{\langle\frac{s_1,s_4}{0.2}\rangle, \langle\frac{s_2}{0.4}\rangle, \langle\frac{s_3}{0.6}\rangle\}$ and
$\mathcal{F}\mathfrak{Nano}Zint(A \vee B) = \{\langle\frac{s_1,s_4}{0.2}\rangle, \langle\frac{s_2}{0.5}\rangle, \langle\frac{s_3}{0.6}\rangle\}$.
Thus $\mathcal{F}\mathfrak{Nano}Zint(A \vee B) \not\leqslant \mathcal{F}\mathfrak{Nano}Zint(A) \vee \mathcal{F}\mathfrak{Nano}Zint(B)$.

(ii) $C = \{\langle\frac{s_1,s_4}{0.1}\rangle, \langle\frac{s_2}{0.5}\rangle, \langle\frac{s_3}{0.7}\rangle\}$ and $D = \{\langle\frac{s_1,s_4}{0.2}\rangle, \langle\frac{s_2}{0.4}\rangle, \langle\frac{s_3}{0.6}\rangle\}$, then
$C \wedge D = \{\langle\frac{s_1,s_4}{0.1}\rangle, \langle\frac{s_2}{0.4}\rangle, \langle\frac{s_3}{0.6}\rangle\}$.
$\mathcal{F}\mathfrak{Nano}Zint(C) = \{\langle\frac{s_1,s_4}{0.1}\rangle, \langle\frac{s_2}{0.5}\rangle, \langle\frac{s_3}{0.7}\rangle\}$,
$\mathcal{F}\mathfrak{Nano}Zint(D) = \{\langle\frac{s_1,s_4}{0.2}\rangle, \langle\frac{s_2}{0.4}\rangle, \langle\frac{s_3}{0.6}\rangle\}$ and
$\mathcal{F}\mathfrak{Nano}Zint(C \wedge D) = \{\langle\frac{s_1,s_4}{0.1}\rangle, \langle\frac{s_2}{0.3}\rangle, \langle\frac{s_3}{0.4}\rangle\}$.
Thus $\mathcal{F}\mathfrak{Nano}Zint(C) \wedge \mathcal{F}\mathfrak{Nano}Zint(D) \not\leqslant \mathcal{F}\mathfrak{Nano}Zint(C \wedge D)$.

Theorem 2.25 Let $(V, \tau_{\mathcal{F}}(F))$ be a $\mathcal{F}\mathfrak{Nano}ts$ and $S \leqslant A$. Then S is a $\mathcal{F}\mathfrak{Nano}Zo$ set iff $S = \mathcal{F}\mathfrak{Nano}\delta\mathcal{S}int(S) \vee \mathcal{F}\mathfrak{Nano}\mathcal{P}int(S)$.

Proof. Let S be a $\mathcal{F}\mathfrak{Nano}Zo$ set. Then $S \leqslant \mathcal{F}\mathfrak{Nano}cl(\mathcal{F}\mathfrak{Nano}\delta int(S)) \vee \mathcal{F}\mathfrak{Nano}int(\mathcal{F}\mathfrak{Nano}cl(S))$ and hence by Proposition 2.17 and Lemma 2.18, $\mathcal{F}\mathfrak{Nano}\delta\mathcal{S}int(S) \vee \mathcal{F}\mathfrak{Nano}\mathcal{P}int(S) = (A \wedge \mathcal{F}\mathfrak{Nano}cl(\mathcal{F}\mathfrak{Nano}\delta int(S))) \vee (A \wedge \mathcal{F}\mathfrak{Nano}int(\mathcal{F}\mathfrak{Nano}cl(S))) = S \wedge (\mathcal{F}\mathfrak{Nano}cl(\mathcal{F}\mathfrak{Nano}\delta int(S)) \vee \mathcal{F}\mathfrak{Nano}int(\mathcal{F}\mathfrak{Nano}cl(S))) = S$. It follows from Proposition 2.17 and Lemma 2.18 that the converse is true. ■

Proposition 2.26 Let $(V, \tau_{\mathcal{F}}(F))$ be a $\mathcal{F}\mathfrak{Nano}ts$ and $S \leqslant A$. Then S is a $\mathcal{F}\mathfrak{Nano}Zc$ set iff $S = \mathcal{F}\mathfrak{Nano}\delta\mathcal{S}cl(S) \wedge \mathcal{F}\mathfrak{Nano}\mathcal{P}cl(S)$.

Theorem 2.27 Assume that S is a subset of the space $(V, \tau_{\mathcal{F}}(F))$. Then

(i) $\mathcal{F}\mathfrak{Nano}Zcl(S) = \mathcal{F}\mathfrak{Nano}\delta\mathcal{S}cl(S) \wedge \mathcal{F}\mathfrak{Nano}\mathcal{P}cl(S)$,
(ii) $\mathcal{F}\mathfrak{Nano}Zint(S) = \mathcal{F}\mathfrak{Nano}\delta\mathcal{S}int(S) \vee \mathcal{F}\mathfrak{Nano}\mathcal{P}int(S)$.

Proof. (i) It is easy to see that $\mathcal{F}\mathfrak{Nano}Zcl(S) \leqslant \mathcal{F}\mathfrak{Nano}\delta\mathcal{S}cl(S) \wedge \mathcal{F}\mathfrak{Nano}\mathcal{P}cl(S)$. Also, $\mathcal{F}\mathfrak{Nano}\delta\mathcal{S}cl(S) \wedge \mathcal{F}\mathfrak{Nano}\mathcal{P}cl(S) = (S \vee \mathcal{F}\mathfrak{Nano}int(\mathcal{F}\mathfrak{Nano}\delta cl(S))) \wedge (S \vee \mathcal{F}\mathfrak{Nano}cl(\mathcal{F}\mathfrak{Nano}int(S))) = S \vee (\mathcal{F}\mathfrak{Nano}int(\mathcal{F}\mathfrak{Nano}\delta cl(S)) \wedge \mathcal{F}\mathfrak{Nano}cl(\mathcal{F}\mathfrak{Nano}int(S)))$.

Since $\mathcal{F}\mathfrak{Nano}Zcl(S)$ is $\mathcal{F}\mathfrak{Nano}Zc$, then $\mathcal{F}\mathfrak{Nano}Zcl(S) \leqslant \mathcal{F}\mathfrak{Nano}int(\mathcal{F}\mathfrak{Nano}\delta cl(\mathcal{F}\mathfrak{Nano}Zcl(S))) \wedge \mathcal{F}\mathfrak{Nano}cl(\mathcal{F}\mathfrak{Nano}int(\mathcal{F}\mathfrak{Nano}Zcl(S))) \geqslant \mathcal{F}\mathfrak{Nano}int(\mathcal{F}\mathfrak{Nano}\delta cl(S)) \wedge \mathcal{F}\mathfrak{Nano}cl(\mathcal{F}\mathfrak{Nano}int(S))$.

Thus, $S \vee (\mathcal{F}\mathfrak{Nano}int(\mathcal{F}\mathfrak{Nano}\delta cl(S)) \wedge \mathcal{F}\mathfrak{Nano}cl(\mathcal{F}\mathfrak{Nano}int(S))) \leqslant S \vee \mathcal{F}\mathfrak{Nano}Zcl(S) = \mathcal{F}\mathfrak{Nano}Zcl(S)$ and hence, $\mathcal{F}\mathfrak{Nano}\delta\mathcal{S}cl(S) \wedge \mathcal{F}\mathfrak{Nano}\mathcal{P}cl(S) \leqslant \mathcal{F}\mathfrak{Nano}Zcl(S)$. So, $\mathcal{F}\mathfrak{Nano}Zcl(S) = \mathcal{F}\mathfrak{Nano}\delta\mathcal{S}cl(S) \wedge \mathcal{F}\mathfrak{Nano}\mathcal{P}cl(S)$.

(ii) As a result of (i). ■

Theorem 2.28 Let S be a $\mathcal{F}subs$ of a space $(V, \tau_{\mathcal{F}}(F))$. Then

(i) S is a $\mathcal{F}\mathfrak{Nano}Zo$ set iff $S = \mathcal{F}\mathfrak{Nano}Zint(S)$,
(ii) S is a $\mathcal{F}\mathfrak{Nano}Zc$ set iff $S = \mathcal{F}\mathfrak{Nano}Zcl(S)$.

Proof. (i) Theorems 2.25 & 2.27 lead to this conclusion. ■

Lemma 2.29 Let S be a $\mathcal{F}subs$ of a $\mathcal{F}\mathfrak{Nano}ts$ $(V, \tau_{\mathcal{F}}(F))$. Then the following assertion is true:

(i) $\mathcal{F}\mathfrak{Nano}\delta\mathcal{P}int(\mathcal{F}\mathfrak{Nano}\mathcal{P}cl(S)) = \mathcal{F}\mathfrak{Nano}\mathcal{P}cl(S) \wedge \mathcal{F}\mathfrak{Nano}int(\mathcal{F}\mathfrak{Nano}\delta cl(S))$,
(ii) $\mathcal{F}\mathfrak{Nano}\delta\mathcal{P}cl(\mathcal{F}\mathfrak{Nano}\mathcal{P}int(S)) = \mathcal{F}\mathfrak{Nano}\mathcal{P}int(S) \vee \mathcal{F}\mathfrak{Nano}cl(\mathcal{F}\mathfrak{Nano}\delta int(S))$.

Proof. (i) By Lemma 2.19, $\mathcal{F}\mathfrak{Nano}\delta\mathcal{P}int(\mathcal{F}\mathfrak{Nano}\mathcal{P}cl(S)) = \mathcal{F}\mathfrak{Nano}\mathcal{P}cl(S) \wedge \mathcal{F}\ \mathfrak{Nano}int(\mathcal{F}\mathfrak{Nano}\delta cl(\mathcal{F}\mathfrak{Nano}\mathcal{P}cl(S))) = \mathcal{F}\mathfrak{Nano}\mathcal{P}cl(S) \wedge \mathcal{F}\mathfrak{Nano}int(\mathcal{F}\mathfrak{Nano}\delta cl(S \vee \mathcal{F}\mathfrak{Nano}int(\mathcal{F}\mathfrak{Nano}cl(S)))) = \mathcal{F}\mathfrak{Nano}\mathcal{P}cl(S) \wedge \mathcal{F}\mathfrak{Nano}int(\mathcal{F}\mathfrak{Nano}\delta cl(S))$.
(ii) As a result of (i). ∎

Proposition 2.30 Let S be a $\mathcal{F}subs$ of a $\mathcal{F}\mathfrak{Nano}ts$ $(V, \tau_{\mathcal{F}}(F))$. Then

(i) $\mathcal{F}\mathfrak{Nano}Zcl(S) = S \vee \mathcal{F}\mathfrak{Nano}\delta\mathcal{P}int(\mathcal{F}\mathfrak{Nano}\mathcal{P}cl(S))$,
(ii) $\mathcal{F}\mathfrak{Nano}Zint(S) = S \wedge \mathcal{F}\mathfrak{Nano}\delta\mathcal{P}cl(\mathcal{F}\mathfrak{Nano}\mathcal{P}int(S))$.

Proof. (i) By Lemma 2.29, $S \vee \mathcal{F}\mathfrak{Nano}\delta\mathcal{P}int(\mathcal{F}\mathfrak{Nano}\mathcal{P}cl(S)) = S \vee (\mathcal{F}\mathfrak{Nano}\mathcal{P}\ cl(S) \wedge \mathcal{F}\mathfrak{Nano}int(\mathcal{F}\mathfrak{Nano}\delta cl(S))) = (S \vee \mathcal{F}\mathfrak{Nano}\mathcal{P}cl(S)) \wedge (S \vee \mathcal{F}\mathfrak{Nano}int(\mathcal{F}\mathfrak{Nano}\ \delta cl(S))) = \mathcal{F}\mathfrak{Nano}\mathcal{P}cl(S) \wedge \mathcal{F}\mathfrak{Nano}\delta Scl(S) = \mathcal{F}\mathfrak{Nano}Zcl(S)$.
(ii) As a result of (i). ∎

Theorem 2.31 Let S be a $\mathcal{F}subs$ of a $\mathcal{F}\mathfrak{Nano}ts$ $(V, \tau_{\mathcal{F}}(F))$. Then the following assertion is true:

(i) S is a $\mathcal{F}\mathfrak{Nano}Zo$ set,
(ii) $S \leqslant \mathcal{F}\mathfrak{Nano}\delta\mathcal{P}cl(\mathcal{F}\mathfrak{Nano}\mathcal{P}int(S))$,
(iii) $\exists\ M \in \mathcal{F}\mathfrak{Nano}\mathcal{P}O(V) \ni M \leqslant S \leqslant \mathcal{F}\mathfrak{Nano}\delta\mathcal{P}cl(M)$,
(iv) $\mathcal{F}\mathfrak{Nano}\delta\mathcal{P}cl(S) = \mathcal{F}\mathfrak{Nano}\delta\mathcal{P}cl(\mathcal{F}\mathfrak{Nano}\mathcal{P}int(S))$.

Proof. (i) ⇒ (ii). Let S be a $\mathcal{F}\mathfrak{Nano}Zo$ set. Then by Theorem 2.28, $S = \mathcal{F}\mathfrak{Nano}Zint(S)$ and by Proposition 2.30, $S = S \wedge \mathcal{F}\mathfrak{Nano}\delta\mathcal{P}cl(\mathcal{F}\mathfrak{Nano}\mathcal{P}int(S))$ and hence, $S \leqslant \mathcal{F}\mathfrak{Nano}\delta\mathcal{P}cl(\mathcal{F}\mathfrak{Nano}\mathcal{P}int(S))$.
(ii) ⇒ (i). Let $S \leqslant \mathcal{F}\mathfrak{Nano}\delta\mathcal{P}cl(\mathcal{F}\mathfrak{Nano}\mathcal{P}int(S))$. Then by Proposition 2.30, $S \leqslant S \wedge \mathcal{F}\mathfrak{Nano}\delta\mathcal{P}cl(\mathcal{F}\mathfrak{Nano}\mathcal{P}int(S)) = \mathcal{F}\mathfrak{Nano}Zint(S)$, and hence $S = \mathcal{F}\mathfrak{Nano}\ Zint(S)$. Thus S is $\mathcal{F}\mathfrak{Nano}Zo$.
(ii) ⇒ (iii). It comes as a result of putting $M = \mathcal{F}\mathfrak{Nano}\mathcal{P}int(S)$.
(iii) ⇒ (ii). Let $\exists\ M \in \mathcal{F}\mathfrak{Nano}\mathcal{P}O(V)$ such that $M \leqslant S \leqslant \mathcal{F}\mathfrak{Nano}\delta\mathcal{P}cl(M)$. Since $M \leqslant S$, then $\mathcal{F}\mathfrak{Nano}\delta\mathcal{P}cl(M) \leqslant \mathcal{F}\mathfrak{Nano}\delta\mathcal{P}cl(\mathcal{F}\mathfrak{Nano}\mathcal{P}int(S))$, therefore $S \leqslant \mathcal{F}\mathfrak{Nano}\delta\mathcal{P}cl(M) \leqslant \mathcal{F}\mathfrak{Nano}\delta\mathcal{P}cl(\mathcal{F}\mathfrak{Nano}\mathcal{P}int(S))$.
(ii) ⇔ (iv). It is clear. ∎

Theorem 2.32 Let S be a $\mathcal{F}subs$ of a $\mathcal{F}\mathfrak{Nano}ts$ $(V, \tau_{\mathcal{F}}(F))$. Then the following assertion is true:

(i) S is a $\mathcal{F}\mathfrak{Nano}Zc$ set,
(ii) $\mathcal{F}\mathfrak{Nano}\delta\mathcal{P}int(\mathcal{F}\mathfrak{Nano}\mathcal{P}cl(S)) \leqslant S$,
(iii) $\exists\ M \in \mathcal{F}\mathfrak{Nano}PC(A) \ni \mathcal{F}\mathfrak{Nano}\delta\mathcal{P}int(M) \leqslant S \leqslant M$,
(iv) $\mathcal{F}\mathfrak{Nano}\delta\mathcal{P}int(S) = \mathcal{F}\mathfrak{Nano}\delta\mathcal{P}int(\mathcal{F}\mathfrak{Nano}\mathcal{P}cl(S))$.

Proposition 2.33 If S is a $\mathcal{F}\mathfrak{Nano}Zo$ set of a $\mathcal{F}\mathfrak{Nano}ts$ $(V, \tau_{\mathcal{F}}(F))$ such that $S \leqslant L \leqslant \mathcal{F}\mathfrak{Nano}\delta\mathcal{P}cl(S)$, then L is $\mathcal{F}\mathfrak{Nano}Zo$.

Proof. It is clear. ∎

Definition 2.34 A set S of a $\mathcal{F}\mathfrak{Nano}ts$ $(V, \tau_{\mathcal{F}}(F))$ is said to be locally $\mathcal{F}\mathfrak{Nano}Zc$ if $S = L \wedge M$, where $L \in \mathcal{F}\mathfrak{Nano}O(V)$ and $M \in \mathcal{F}\mathfrak{Nano}ZC(V)$.

Theorem 2.35 Let S be a $\mathcal{F}subs$ of a $\mathcal{F}\mathfrak{Nano}ts$ $(V, \tau_{\mathcal{F}}(F))$. Then S is locally $\mathcal{F}\mathfrak{Nano}Zc$ iff $S = L \wedge \mathcal{F}\mathfrak{Nano}Zcl(S)$.

Proof. Only prove first part. Since S is a locally $\mathcal{F}\mathfrak{Nano}Zc$ set, then $S = L \wedge M$, where $L \in \mathcal{F}\mathfrak{Nano}L(V)$ and $M \in \mathcal{F}\mathfrak{Nano}ZC(V)$ and hence $S \leqslant \mathcal{F}\mathfrak{Nano}Zcl(S) \leqslant \mathcal{F}\mathfrak{Nano}Zcl(M) = M$. Thus $S \leqslant L \wedge \mathcal{F}\mathfrak{Nano}Zcl(S) \leqslant L \wedge \mathcal{F}\mathfrak{Nano}Zcl(M) = S$.

Therefore $S = L \wedge \mathcal{F}\mathfrak{Nano}Zcl(S)$. ■

Theorem 2.36 Let S be a locally $\mathcal{F}\mathfrak{Nano}Zc$ set of a space $(V, \tau_{\mathcal{F}}(F))$. Then the following assertion is true:

(i) $\mathcal{F}\mathfrak{Nano}Zcl(S) - S$ is a $\mathcal{F}\mathfrak{Nano}Zc$ set,
(ii) $(S \vee (1_{\mathcal{F}} - \mathcal{F}\mathfrak{Nano}Zcl(S)))$ is a $\mathcal{F}\mathfrak{Nano}Zo$ set,
(iii) $S \leqslant \mathcal{F}\mathfrak{Nano}Zint(S \vee (1_{\mathcal{F}} - \mathcal{F}\mathfrak{Nano}Zcl(S)))$.

Proof. (i) If S is a locally $\mathcal{F}\mathfrak{Nano}Zc$ set, then there exists a $\mathcal{F}\mathfrak{Nano}o$ set L such that $S = L \wedge \mathcal{F}\mathfrak{Nano}Zcl(S)$. Hence, $\mathcal{F}\mathfrak{Nano}Zcl(S) - S = \mathcal{F}\mathfrak{Nano}Zcl(S) - (L \wedge \mathcal{F}\mathfrak{Nano}Zcl(S)) = \mathcal{F}\mathfrak{Nano}Zcl(S) \wedge (1_{\mathcal{F}} - (L \wedge \mathcal{F}\mathfrak{Nano}Zcl(S))) = \mathcal{F}\mathfrak{Nano}Zcl(S) \wedge (1_{\mathcal{F}} - L) \vee (1_{\mathcal{F}} - \mathcal{F}\mathfrak{Nano}Zcl(S)) = \mathcal{F}\mathfrak{Nano}Zcl(S) \wedge (1_{\mathcal{F}} - L)$ which is $\mathcal{F}\mathfrak{Nano}Zc$.

(ii) Since $\mathcal{F}\mathfrak{Nano}Zcl(S) - S$ is $\mathcal{F}\mathfrak{Nano}Zc$, then $1_{\mathcal{F}} - (\mathcal{F}\mathfrak{Nano}Zcl(S) - S)$ is a $\mathcal{F}\mathfrak{Nano}Zo$ set. Since $1_{\mathcal{F}} - (\mathcal{F}\mathfrak{Nano}Zcl(S) - S) = ((1_{\mathcal{F}} - \mathcal{F}\mathfrak{Nano}Zcl(S)) \vee (1_{\mathcal{F}} \wedge S)) = (S \vee (1_{\mathcal{F}} - \mathcal{F}\mathfrak{Nano}Zcl(S)))$, then $S \vee (1_{\mathcal{F}} - \mathcal{F}\mathfrak{Nano}Zcl(S))$ is $\mathcal{F}\mathfrak{Nano}Zo$.

(iii) As a result of (ii). ■

Definition 2.37 A $\mathcal{F}s$ S of a space $(V, \tau_{\mathcal{F}}(F))$ is said to be fuzzy nano $D(c, z)$ (briefly, $\mathcal{F}\mathfrak{Nano}D(c, z)$) iff $\mathcal{F}\mathfrak{Nano}int(S) = \mathcal{F}\mathfrak{Nano}Zint(S)$.

Remark 5 *It's worth noting that the concepts of $\mathcal{F}\mathfrak{Nano}Zo$ and $\mathcal{F}\mathfrak{Nano}D(c, z)$ are not independent, as shown in the example below.*

***Example* 2.38** In Example 2.5, the $\mathcal{F}\mathfrak{Nano}$ sets

(i) $\left\{\left\langle\frac{s_1, s_4}{0.3}\right\rangle, \left\langle\frac{s_2}{0.5}\right\rangle, \left\langle\frac{s_3}{0.7}\right\rangle\right\}$ is a $\mathcal{F}\mathfrak{Nano}Zos$ but not $\mathcal{F}\mathfrak{Nano}D(c, z)$.
(ii) $\left\{\left\langle\frac{s_1, s_4}{0.1}\right\rangle, \left\langle\frac{s_2}{0.3}\right\rangle, \left\langle\frac{s_3}{0.5}\right\rangle\right\}$ is a $\mathcal{F}\mathfrak{Nano}D(c, z)$ but not $\mathcal{F}\mathfrak{Nano}Zos$.

Theorem 2.39 Let S be a $\mathcal{F}subs$ of $\mathcal{F}\mathfrak{Nano}ts$ $(V, \tau_{\mathcal{F}}(F))$. Then the following will suffice:

(i) S is an $\mathcal{F}\mathfrak{Nano}o$ set,
(ii) S is $\mathcal{F}\mathfrak{Nano}Zo$ and $\mathcal{F}\mathfrak{Nano}D(c, z)$.

3. Conclusion

We investigated the properties of a new class of sets termed $\mathcal{F}\mathfrak{Nano}Zo$ sets in $\mathcal{F}\mathfrak{Nano}ts$ in this article. Fuzzy nano Z-interior, fuzzy nano Z-closure and their relationships with well-known fuzzy sets were also discussed. Fuzzy nano Z continuous functions, fuzzy nano Z open mapping, fuzzy nano Z closed mapping, and fuzzy nano Z homeomorphic functions can all be added in the future.

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