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Best proximity of proximal $\mathcal{F}^*$-weak contraction

M. Salamatbakhsh[a], R. H. Haghi[a,*], K. Fallahi[a]
[a] *Department of Mathematics, Payame Noor University, Tehran, Iran.*

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Abstract. Best proximity point theorems for self-mappings were investigated with different conditions on spaces for contraction mappings. In this paper, we prove best proximity point theorems for proximal $\mathcal{F}^*$-weak contraction mappings.

Keywords: Best Proximity point, proximal $\mathcal{F}^*$-weak contraction, approximatively compact, cyclically Cauchy sequence, uniform approximation, $\mathcal{S}$-approximation, quasi-continuous.

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1. Introduction and preliminaries

Numerous of real life problems such as system of linear or algebraic equations, ordinary or partial differential equations, etc can be framed as linear or nonlinear equations of the form $\mathcal{S}a = a$. In this case a is called a fixed point of $\mathcal{S}$. To study the existence of fixed points for discontinuous mappings, Kannan [7] introduced a weaker contraction condition and proved a very interesting fixed point result. Rhoades [11] compared various contractive definitions and he showed that though most of the contractive definitions do not force the mapping to be continuous on the entire domain, all of them force the mapping to be continuous at the fixed point. Motivated by his observations, Rhoades [12] formulated an interesting open question whether there exists a contractive definition which is strong enough to ensure the existence and uniqueness of a fixed point which does not force the mapping to be continuous at the fixed point.For more details, we refer to [1, 2, 9] and reference contained therein. The first answer of the this open question

*Corresponding author.
E-mail address: mahdi.salamatbakhsh@gmail.com (M. Salamatbakhsh); Robab.haghi@gmail.com (R. H. Haghi); fallahi1361@gmail.com (K. Fallahi).

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appeared after more than a decade by Pant [10]. On the other hand, Wardowski [19] introduced a new class of functions and denoted them by $\mathcal{F}-$contractions mappings. In recent years, the concept of $\mathcal{F}-$contractions has attracted the attention of several researchers. By now, there exists a considerable literature in enriching this idea (see [3, 6] and references therein).

All maps have not fixed point. In this case, one tries to gain an approximate solutions a subject to the condition that the distance between a and $\mathcal{S}a$ is minimum. Indeed, best proximity point theorems explore the existence of such optimal approximate solutions, known as best proximity points of map or multifunction. Then a is the best proximity of $\mathcal{S}:\mathcal{A}\to\mathcal{B}$ if $\rho(a,\mathcal{S}a)=\rho(\mathcal{A},\mathcal{B})$.

A best proximity point theorem for non-self-contractions has been investigated in [15]. In [13] Sadiq Basha defined proximal contraction mappings $\mathcal{S}:\mathcal{A}\to\mathcal{B}$ and proved the best proximity point for these mappings.

In this paper we introduce proximal $\mathcal{F}^*$-weak contraction mappings of first and second kind, then we proved best proximity point theorems for such contractions. Throughout this paper, let $(\mathcal{M},\rho)$ be a metric space and $\mathcal{A}$ and $\mathcal{B}$ be nonempty subsets of $\mathcal{M}$. We recall the following notations, which will be used in the sequel.

$$\begin{aligned}\rho(\mathcal{A},\mathcal{B}) &:= \inf\{\rho(a,b) : a\in\mathcal{A} \text{ and } b\in\mathcal{B}\},\\ \mathcal{A}_0 &:= \{a\in\mathcal{A} : \rho(a,b)=\rho(\mathcal{A},\mathcal{B}) \text{ for some } b\in\mathcal{B}\},\\ B_0 &:= \{b\in\mathcal{B} : \rho(a,b)=\rho(\mathcal{A},\mathcal{B}) \text{ for some } a\in\mathcal{A}\}.\end{aligned}$$

In [8], sufficient conditions are provided to guarantee the non-empties of $\mathcal{A}_0$ and $\mathcal{B}_0$. Also, in the setting of normed linear spaces, if $\mathcal{A}$ and $\mathcal{B}$ are closed subsets such that $\rho(\mathcal{A},\mathcal{B})>0$, then $\mathcal{A}_0$ and $\mathcal{B}_0$ are contained in the boundaries of $\mathcal{A}$ and $\mathcal{B}$, respectively [17].

Definition 1.1 [13] The set $\mathcal{B}$ is said to be approximatively compact with respect to $\mathcal{A}$ if every sequence $\{b_n\}$ of $\mathcal{B}$ satisfying the condition that $\rho(a,b_n)\to\rho(a,B)$ for some $a\in\mathcal{A}$ has a convergent subsequence.

It is trivial to note that every set is approximatively compact with respect to itself, and that every compact set is approximatively compact with respect to any arbitrary set. Further, $\mathcal{A}_0$ and $\mathcal{B}_0$ are nonempty if $\mathcal{A}$ is compact and $\mathcal{B}$ is approximatively compact with respect to $\mathcal{A}$.

Definition 1.2 [13] A point $a^*\in\mathcal{A}$ is called a best proximity point of map $\mathcal{S}:\mathcal{A}\to\mathcal{B}$ if it satisfies the condition that $\rho(a^*,\mathcal{S}a^*)=\rho(\mathcal{A},\mathcal{B})$.

It is apparent that a best proximity point serves as a global minimizer of $a\to\rho(a,\mathcal{S}a)$, Since we have $\rho(a,\mathcal{S}a)\geqslant\rho(\mathcal{A},\mathcal{B})$ for all $a\in\mathcal{A}$. As a result, a best proximity point represents an optimal approximate solution of $a=\mathcal{S}a$ in the sense that it is an approximate solution of $a=\mathcal{S}a$ with the least possible error.

Definition 1.3 [16] Let $\{a_n\}$ be a sequence in $\mathcal{A}$ and $\{b_n\}$ be a sequence in $\mathcal{B}$. Then, the sequence $\{(a_n,b_n)\}$ in $\mathcal{A}\times\mathcal{B}$ is said to be a cyclically Cauchy sequence if and only if for every $\epsilon>0$, there exits a positive integer N such that $\rho(a_m,b_n)<\rho(\mathcal{A},\mathcal{B})+\epsilon$ for all $m,n\geqslant N$.

The sequence $\{a_n\}$ in $\mathcal{A}$ is a Cauchy sequence if and only if the sequence $\{(a_n,a_n)\}$ is a cyclically Cauchy sequence in $\mathcal{A}\times\mathcal{A}$.

Definition 1.4 [16] Let $\{a_n\}$ be a sequence in $\mathcal{A}$ and $\{b_n\}$ be a sequence in $\mathcal{B}$. Then,

the sequence $\{(a_n, b_n)\}$ in $\mathcal{A} \times \mathcal{B}$ is said to be a fairly Cauchy sequence if and only if the following conditions are satisfied:
(i) $\{(a_n, b_n)\}$ is a cyclically Cauchy sequence;
(ii) $\{a_n\}$ and $\{b_n\}$ are Cauchy sequence.

Definition 1.5 [16] The pair $(\mathcal{A}, \mathcal{B})$ is called a fairly complete space if and only if for every fairly Cauchy sequence $\{(a_n, b_n)\}$ in $\mathcal{A} \times \mathcal{B}$, the sequences $\{a_n\}$ and $\{b_n\}$ are convergent in $\mathcal{A}$ and $\mathcal{B}$, respectively.

Definition 1.6 [14] A set $\mathcal{A}$ is said to have uniform approximation in $\mathcal{B}$ if and only if, for given $\epsilon > 0$, there exists $\delta > 0$ such that

$$\begin{cases} \rho(a_1, b_1) = \rho(\mathcal{A}, \mathcal{B}) \\ \rho(a_2, b_2) = \rho(\mathcal{A}, \mathcal{B}) \\ \rho(a_1, a_2) < \delta \end{cases} \text{implies } \rho(b_1, b_2) < \epsilon$$

for all $a_1, a_2 \in A$ and $b_1, b_2 \in \mathcal{B}$.

Definition 1.7 [13] Let $\mathcal{S} : \mathcal{A} \to \mathcal{B}$ be a map. Then, $\mathcal{A}$ is said to have uniform $\mathcal{S}$-approximation in $\mathcal{B}$ if, for given $\epsilon > 0$, there exists $\delta > 0$ such that

$$\begin{cases} \rho(u_1, \mathcal{S}a_1) = \rho(\mathcal{A}, \mathcal{B}) \\ \rho(u_2, \mathcal{S}a_2) = \rho(\mathcal{A}, \mathcal{B}) \\ \rho(u_1, u_2) < \delta \end{cases} \text{implies } \rho(\mathcal{S}a_1, \mathcal{S}a_2) < \epsilon$$

for all $a_1, a_2, u_1, u_2 \in \mathcal{A}$.

Definition 1.8 Given non-empty subsets $\mathcal{A}$ and $\mathcal{B}$ of a metric space, a map $\mathcal{S} : \mathcal{A} \to \mathcal{B}$ is said to be a proximally quasi-continuous if

$$\begin{cases} \rho(u_n, \mathcal{S}a_n) = \rho(\mathcal{A}, \mathcal{B}) \\ \rho(u, \mathcal{S}a) = \rho(\mathcal{A}, \mathcal{B}) \\ a_n \to a \end{cases} \quad \text{implies } u_{n_k} \to u \text{ for some subsequence } u_{n_k} \text{ of } u_n$$

for $a, u \in \mathcal{A}$ and for all sequences $\{u_n\}$ and $\{a_n\}$ in $\mathcal{A}$.

Definition 1.9 [19] Let $\mathcal{F} : (0, +\infty) \to \mathbb{R}$ be a mapping satisfying:
$\mathcal{F}_\mathbf{1}$: $\mathcal{S}$ is strictly increasing, that is, $\alpha < \beta$ implies $\mathcal{F}(\alpha) < \mathcal{F}(\beta)$ for all $\alpha, \beta \in (0, +\infty)$,
$\mathcal{F}_\mathbf{2}$: For every sequence $\{\alpha_n\}$ in $(0, +\infty)$ we have

$$\lim_{n\to+\infty} \alpha_n = 0 \qquad \text{if and only if} \qquad \lim_{n\to+\infty} \mathcal{F}(\alpha_n) = -\infty$$

$\mathcal{F}_\mathbf{3}$: There exists a number $k \in (0, 1)$ such that $\lim_{\alpha\to 0} \alpha^k \mathcal{F}(\alpha) = 0$

We denote with $\mathbf{\Psi}$ the family of all functions $\mathcal{F}$ which satisfy the conditions $\mathcal{F}_\mathbf{1} - \mathcal{F}_\mathbf{3}$.

***Example* 1.10** Let $\mathcal{F}_i : (0, +\infty) \to \mathbb{R}$, where $i = 1, 2, 3, 4$ be defined by

$$\mathcal{F}_1(\alpha) = \frac{1}{\sqrt{\alpha}}, \quad \mathcal{F}_2(\alpha) = \ln(\alpha), \quad \mathcal{F}_3(\alpha) = \ln(\alpha) + \alpha, \quad \mathcal{F}_4(\alpha) = \ln(\alpha^2 + \alpha).$$

Then $\mathcal{F}_\mathbf{1}, \mathcal{F}_\mathbf{2}, \mathcal{F}_\mathbf{3}, \mathcal{F}_\mathbf{4} \in \mathbf{\Psi}$.

It was remarked in [19] that the monotonicity of $\mathcal{F}$ implies that every $\mathcal{F}$-contraction mapping is contractive and hence continuous. Secelean [18] observed that the continuity of any $\mathcal{F}$-contraction mapping can be obtained from the condition $\mathcal{F}_{\mathbf{2}}$. Wardowski1 and Dung [20] used the same class of auxiliary functions to introduce the notion of $\mathcal{F}-$weak contractions that we denote it with $\mathbf{\Psi}^*$ that is the family of all functions $\mathcal{F}$ such that satisfy the conditions $\mathcal{F}_{\mathbf{2}}$. Obviously, $\mathbf{\Psi}^* \subseteq \mathbf{\Psi}$. However, the converse inclusion is not true in general as shown in the following examples:

Example **1.11** Let $\mathcal{F} : (0, +\infty) \to \mathbb{R}$ be given by $\mathcal{F}(\alpha) = ln(\frac{\alpha}{2}) + \sin(\alpha)$. It is clear that $\mathcal{F}$ satisfies $\mathcal{F}_{\mathbf{2}}$. However, it does not satisfy $\mathcal{F}_{\mathbf{1}}$.

Example **1.12** Let $\mathcal{F} : (0, +\infty) \to \mathbb{R}$ be given by $\mathcal{F}(\alpha) = cos(\alpha) - \frac{1}{\alpha}$. It is clear that $\mathcal{F}$ satisfies $\mathcal{F}_{\mathbf{2}}$. However, it does not satisfy $\mathcal{F}_{\mathbf{1}}$ and $\mathcal{F}_{\mathbf{3}}$.

Next, we introduce the notion of proximal $\mathcal{F}^*-$weak contraction mappings, which runs as follows.

Definition 1.13 A mapping $\mathcal{S} : \mathcal{A} \to \mathcal{B}$ is called a proximal $\mathcal{F}^*-$weak contraction of the first kind if there exists $\mathcal{F} \in \mathbf{\Psi}^*$ and $\tau > 0$ such that

$$\rho(u_1, \mathcal{S}a_1) = \rho(u_2, \mathcal{S}a_2) = \rho(\mathcal{A}, \mathcal{B}) \quad \text{implies} \quad \tau + \mathcal{F}(\rho(u_1, u_2)) \leqslant \mathcal{F}(\rho(a_1, a_2)),$$

where $a_1, a_2, u_1, u_2 \in \mathcal{A}$ and $a_1 \neq a_2, u_1 \neq u_2$.

Definition 1.14 Let $\mathcal{A}, \mathcal{B}$ be nonempty subsets of a metric space $\mathcal{M}$. $\mathcal{S} : \mathcal{A} \to \mathcal{B}$ is said to be a proximal $\mathcal{F}^*-$weak contraction of the second kind if and only if there exists $\mathcal{F} \in \mathbf{\Psi}^*$ and $\tau > 0$ such that

$$\begin{cases} \rho(u_1, \mathcal{S}a_1) = \rho(\mathcal{A}, \mathcal{B}) \\ \rho(u_2, \mathcal{S}a_2) = \rho(\mathcal{A}, \mathcal{B}) \end{cases} \quad \text{implies} \quad \tau + \mathcal{F}(\rho(\mathcal{S}u_1, \mathcal{S}u_2)) \leqslant \mathcal{F}(\rho(\mathcal{S}a_1, \mathcal{S}a_2)),$$

where $a_1, a_2, u_1, u_2 \in \mathcal{A}$ and $\mathcal{S}a_1 \neq \mathcal{S}a_2, \mathcal{S}u_1 \neq \mathcal{S}u_2$.

Definition 1.15 A mapping $\mathcal{S} : \mathcal{A} \to \mathcal{B}$ is said to be a strong proximal $\mathcal{F}^*-$weak contraction of the second kind if and only if the following conditions are satisfied:
(a) $\mathcal{S}$ is a proximally quasi-continuous,
(b) $\mathcal{S}$ is a proximal $\mathcal{F}^*-$weak contraction of the second kind.

Lemma 1.16 [4] Let $\{a_n\}$ be a sequence in $\mathcal{M}$. If $\{a_n\}$ is not a Cauchy sequence, then there exist an $\epsilon > 0$ and two sequences $\{a_{n(k)}\}$ and $\{a_{m(k)}\}$ of $\{a_n\}$ such that for all $k \geqslant 1$ and $m(k) < n(k)$,

$$\rho(a_{m(k)}, a_{n(k)}) \geqslant \epsilon \qquad \text{and} \qquad \rho(a_{m(k)}, a_{n(k)-1}) < \epsilon,$$

furthermore, if $\lim_{n\to+\infty} \rho(a_n, a_{n+1}) = 0$, then

$$\lim_{k\to+\infty} \rho(a_{m(k)}, a_{n(k)}) = \lim_{k\to+\infty} \rho(a_{m(k)-1}, a_{n(k)-1}) = \epsilon.$$

2. Main Results

Now, we are ready to state and prove our main results.

Lemma 2.1 Let $\mathcal{A}$ and $\mathcal{B}$ be non-empty subsets of a metric space such that $\mathcal{A}_0$ and $\mathcal{B}_0$ are non-empty and $\mathcal{S} : \mathcal{A} \to \mathcal{B}$ be a map. Also, assume that $\mathcal{S}(a) \in \mathcal{B}_0$ for any $a \in \mathcal{A}_0$. Then there exists a sequence $\{a_n\}$ in $\mathcal{A}_0$ such that $\rho(a_{n+1}, \mathcal{S}a_n) = \rho(\mathcal{A}, \mathcal{B})$ for all $n \in \mathbb{N}$.

Proof. Let $a_0 \in \mathcal{A}_0$. By assumption $\mathcal{S}(a_0) \in \mathcal{B}_0$. Therefore there exists $a_1 \in \mathcal{A}$ such that $\rho(a_1, \mathcal{S}(a_0)) = \rho(\mathcal{A}, \mathcal{B})$. Also, we have $a_1 \in \mathcal{A}_0$. By assumption $\mathcal{S}(a_1) \in \mathcal{B}_0$. Therefore, there exists $a_2 \in \mathcal{A}$ such that $\rho(a_2, \mathcal{S}(a_1)) = \rho(\mathcal{A}, \mathcal{B})$. We have $a_2 \in \mathcal{A}_0$. By repeating the same process, we can make sequence $\{a_n\}$ in $\mathcal{A}$ such that $\rho(a_{n+1}, \mathcal{S}a_n) = \rho(\mathcal{A}, \mathcal{B})$ for all $n \in \mathbb{N}$. ■

Theorem 2.2 Let $(\mathcal{M}, \rho)$ be a complete metric space. Suppose that the following conditions are satisfied:
(i) $\mathcal{A}, \mathcal{B}$ are nonempty subsets of $\mathcal{M}$ and $\mathcal{A}$ is closed;
(ii) $\mathcal{B}$ is approximatively compact with respect to $\mathcal{A}$;
(iii) A_0 is nonempty;
(iv) $\mathcal{S} : \mathcal{A} \to \mathcal{B}$ is a proximal $\mathcal{F}^*-$weak contraction of the first kind;
(v) $\mathcal{S}(\mathcal{A}_0) \subseteq \mathcal{B}_0$;
(vi) $\mathcal{F}$ is continuous.
Then there exists a unique element $a \in \mathcal{A}$ such that $\rho(a, \mathcal{S}a) = \rho(\mathcal{A}, \mathcal{B})$. Further, for any fixed $a_0 \in \mathcal{A}_0$, the sequence $\{a_n\}$ defined by $\rho(a_{n+1}, \mathcal{S}a_n) = \rho(\mathcal{A}, \mathcal{B})$ is convergent to a.

Proof. Let $a_0 \in \mathcal{A}_0$. By using Lemma 2.1, we get the sequence $\{a_n\}$ in $\mathcal{A}_0$ such that, for any $n \in \mathbb{N}$, we have

$$\rho(a_{n+1}, \mathcal{S}a_n) = \rho(a_{n+2}, \mathcal{S}a_{n+1}) = \rho(\mathcal{A}, \mathcal{B}).$$

By (iv), for any $n \in \mathbb{N}$, we get

$$\tau + \mathcal{F}(\rho(a_{n+2}, a_{n+1})) \leqslant \mathcal{F}(\rho(a_{n+1}, a_n)). \tag{1}$$

We are going to show that $\lim_{n\to+\infty} \rho(a_{n+1}, a_n) = 0$. By (1), we have

$$\mathcal{F}(\rho(a_{n+1}, a_n)) \leqslant \mathcal{F}(\rho(a_n, a_{n-1})) - \tau.$$

By repeating this process, for any $n \in \mathbb{N}$, we get

$$\mathcal{F}(\rho(a_{n+1}, a_n)) \leqslant \mathcal{F}(\rho(a_1, a_0)) - n\tau.$$

It follows that $\lim_{n\to+\infty} \mathcal{F}(\rho(a_{n+1}, a_n)) = -\infty$. Consequently

$$\lim_{n\to+\infty} \rho(a_{n+1}, a_n) = 0. \tag{2}$$

Now, we claim that $\{a_n\}$ is a Cauchy sequence. If not, due to Lemma 1.16 and (2), there exist $\epsilon > 0$ and two sub-sequences $\{a_{m(k)}\}$ and $\{a_{n(k)}\}$ of $\{a_n\}$ such that for all $k \geqslant 1$ and $m(k) < n(k)$,

$$\rho(a_{n(k)-1}, a_{m(k)}) < \epsilon \leqslant \rho(a_{n(k)}, a_{m(k)})$$

and

$$\lim_{k\to+\infty} \rho(a_{n(k)}, a_{m(k)}) = \lim_{k\to+\infty} \rho(a_{n(k)-1}, a_{m(k)-1}) = \epsilon. \tag{3}$$

Since

$$\rho(a_{n(k)}, \mathcal{S}a_{n(k)-1}) = \rho(a_{m(k)}, \mathcal{S}a_{m(k)-1}) = \rho(\mathcal{A}, \mathcal{B}),$$

then by applying (iv), we have

$$\tau + \mathcal{F}(\rho(a_{n(k)}, a_{m(k)})) \leqslant \mathcal{F}(\rho(a_{n(k)-1}, a_{m(k)-1})). \tag{4}$$

As $\mathcal{F}$ is continuous, so on letting $k \to +\infty$ in (4) and using (3), we obtain $\tau+\mathcal{F}(\epsilon) \leqslant \mathcal{F}(\epsilon)$, which is a contradiction. Hence, $\{a_n\}$ is a Cauchy sequence. Since $\mathcal{M}$ is a complete metric space and $\mathcal{A}$ is closed, there exists $a \in \mathcal{A}$ such that $a_n \to a$ as $n \to +\infty$. It is easy to prove that

$$\begin{aligned}\rho(a, \mathcal{B}) &\leqslant \rho(a, \mathcal{S}a_n) \leqslant \rho(a, a_{n+1}) + \rho(a_{n+1}, \mathcal{S}a_n)\\ &= \rho(a, a_{n+1}) + \rho(\mathcal{A}, \mathcal{B}) \leqslant \rho(a, a_{n+1}) + \rho(a, B).\end{aligned} \tag{5}$$

Therefore, $\rho(a, \mathcal{S}a_n) \to \rho(a, B)$ as $n \to +\infty$. By (ii), $\{\mathcal{S}a_n\}$ has a convergent subsequence to $b \in B$. Then $\rho(a, b) = \rho(\mathcal{A}, \mathcal{B})$ and so $a \in \mathcal{A}_0$. Therefore, $\mathcal{S}a \in \mathcal{B}_0$. We claim that $\rho(a, \mathcal{T}a) = \rho(\mathcal{A}, \mathcal{B})$, if not, there exists $c \in \mathcal{A}$ such that $\rho(c, \mathcal{S}a) = \rho(\mathcal{A}, \mathcal{B})$. By (iv), we get

$$\tau + \mathcal{F}(\rho(a_{n+1}, c)) \leqslant \mathcal{F}(\rho(a, a_n)).$$

Since $\mathcal{F}$ is a continuous map, then by taking $n \to +\infty$ we get $\tau = 0$ that is a contradiction. Then $\rho(a, \mathcal{S}a) = \rho(\mathcal{A}, \mathcal{B})$. Now, assume that a^* is an another best proximity point of $\mathcal{S}$ (then $\rho(a, a^*) > 0$). We have

$$\rho(a, \mathcal{S}a) = \rho(a^*, \mathcal{S}a^*) = \rho(\mathcal{A}, \mathcal{B}).$$

By (iv), we get

$$\tau + \mathcal{F}(\rho(a, a^*)) \leqslant \mathcal{F}(\rho(a, a^*))$$

That is a contradiction. Then $\rho(a, a^*) = 0$ and so $a = a^*$. ■

Theorem 2.3 Let $(\mathcal{M}, \rho)$ be a complete metric space. Suppose that the following conditions are satisfied:
(i) $\mathcal{A}, \mathcal{B}$ are nonempty closed subsets of $\mathcal{M}$;
(ii) $\mathcal{B}$ is approximatively compact with respect to $\mathcal{A}$;
(iii) $\mathcal{A}_0$ is nonempty;
(iv) $\mathcal{S} : \mathcal{A} \to \mathcal{B}$ is a strong proximal $\mathcal{F}^*-$weak contraction of the second kind;
(v) $\mathcal{S}(\mathcal{A}_0) \subseteq \mathcal{B}_0$;
(vi) $\mathcal{A}$ is continuous.
Then there exists a unique element $a \in \mathcal{A}$ such that $\rho(a, \mathcal{S}a) = \rho(\mathcal{A}, \mathcal{B})$. Further, for any fixed $a_0 \in \mathcal{A}_0$, the sequence $\{a_n\}$ defined by $\rho(a_{n+1}, \mathcal{S}a_n) = \rho(\mathcal{A}, \mathcal{B})$ is convergent to a.

Proof. Let $a_0 \in \mathcal{A}_1$. By using Lemma 2.1 we get a sequence $\{a_n\}$ in $\mathcal{A}_0$ such that, for any $n \in \mathbb{N}$, we have

$$\rho(a_{n+1}, \mathcal{S}a_n) = \rho(a_{n+1}, \mathcal{S}a_n) = \rho(\mathcal{A}, \mathcal{B}).$$

By (iv), for any $n \in \mathbb{N}$, we get

$$\tau + \mathcal{F}(\rho(\mathcal{S}a_{n+2}, \mathcal{S}a_{n+1})) \leqslant \mathcal{F}(\rho(\mathcal{S}a_{n+1}, \mathcal{S}a_n)). \tag{6}$$

We are going to show that $\lim\limits_{n\to+\infty} \rho(\mathcal{S}a_{n+1}, \mathcal{S}a_n) = 0$. By (6), for any $n \in \mathbb{N}$, we have

$$\mathcal{F}(\rho(\mathcal{S}a_{n+1}, \mathcal{S}a_n)) \leqslant \mathcal{F}(\rho(\mathcal{S}a_n, \mathcal{S}a_{n-1})) - \tau.$$

With repeating this process, for any $n \in \mathbb{N}$, we get

$$\mathcal{F}(\rho(\mathcal{S}a_{n+1}, \mathcal{S}a_n)) \leqslant \mathcal{F}(\rho(\mathcal{S}a_1, \mathcal{S}a_0)) - n\tau.$$

It follows that $\lim\limits_{n\to+\infty} F(\rho(\mathcal{S}a_{n+1}, \mathcal{S}a_n)) = -\infty$ and consequently, we have

$$\lim_{n\to+\infty} \rho(\mathcal{S}a_{n+1}, \mathcal{S}a_n) = 0. \tag{7}$$

Now, we claim that $\{\mathcal{S}a_n\}$ is a Cauchy sequence. If not, due to Lemma 1.16 and (7), there exist $\epsilon > 0$ and two sub-sequences $\{\mathcal{S}a_{m(k)}\}$ and $\{\mathcal{S}a_{n(k)}\}$ of $\{\mathcal{S}a_n\}$ such that

$$\rho(\mathcal{S}a_{m(k)}, a_{n(k1}) < \epsilon = \rho(\mathcal{S}a_{n(k)}, \mathcal{S}a_{m(k)})$$

for all $k \geqslant 1$ and $m(k) < n(k)$, and

$$\lim_{k\to+\infty} \rho(\mathcal{S}a_{n(k)}, \mathcal{S}a_{m(k)}) = \lim_{k\to+\infty} \rho(\mathcal{S}a_{n(k)-1}, \mathcal{S}a_{m(k)-1}) = \epsilon. \tag{8}$$

Now, in view of (8), there exists $N \in \mathbb{N}$ such that $\rho(a_{n(k)}, a_{m(k)}) > 0$ for all $k \geqslant N$. Since

$$\rho(a_{n(k)}, \mathcal{S}a_{n(k)-1}) = \rho(a_{m(k)}, \mathcal{S}a_{m(k)-1}) = \rho(\mathcal{A}, \mathcal{B}),$$

then by applying (iv), we have

$$\tau + \mathcal{F}(\rho(\mathcal{S}a_{n(k)}, \mathcal{S}a_{m(k)})) \leqslant \mathcal{F}(\rho(\mathcal{S}a_{n(k)-1}, \mathcal{S}a_{m(k)-1})). \tag{9}$$

As $\mathcal{F}$ is continuous, so on letting $k \to +\infty$ in (9) and using (8), we obtain $\tau + \mathcal{F}(\epsilon) \leqslant \mathcal{F}(\epsilon)$, which is a contradiction. Hence, $\{\mathcal{S}a_n\}$ is a Cauchy sequence. Since $\mathcal{M}$ is a complete metric space and $\mathcal{B}$ is closed, there exists $b \in \mathcal{B}$ such that $\mathcal{S}a_n \to b$ as $n \to +\infty$. Also, we have

$$\begin{aligned} \rho(b, \mathcal{A}) \leqslant \rho(b, a_{n+1}) &\leqslant \rho(b, \mathcal{S}a_n) + \rho(a_{n+1}, \mathcal{S}a_n) \\ &= \rho(b, \mathcal{S}a_n) + \rho(\mathcal{A}, \mathcal{B}) \\ &\leqslant \rho(b, \mathcal{S}a_n) + \rho(b, \mathcal{A}). \end{aligned} \tag{10}$$

Then $\rho(b, a_{n+1}) \to \rho(b, \mathcal{A})$ as $n \to +\infty$. By (ii), $\{a_n\}$ has a convergent subsequence to $a \in \mathcal{A}$. Therefore, $\rho(a, b) = \rho(\mathcal{A}, b)$. By (10), we have

$$\rho(\mathcal{A}, \mathcal{B}) \leqslant \rho(b, \mathcal{A}) \leqslant \rho(b, a_{n+1}) \leqslant \rho(y, \mathcal{S}a_n) + \rho(a_{n+1}, \mathcal{S}a_n).$$

Then

$$\rho(\mathcal{A}, \mathcal{B}) \leqslant \rho(b, a_{n+1}) \leqslant \rho(b, \mathcal{S}a_n) + \rho(\mathcal{A}, \mathcal{B}).$$

Thus, $\rho(a, b) = \rho(\mathcal{A}, \mathcal{B})$ and so we have $a \in \mathcal{A}_0$. By (v), there exists $c \in A$ such that $\rho(c, \mathcal{S}a) = \rho(\mathcal{A}, \mathcal{B})$. Moreover, $\{a_n\}$ has a convergent subsequence like $\{a_{n_k}\}$ such that $a_{n_k} \to a$ as $k \to +\infty$. So we have

$$\rho(a_{n_k+1}, \mathcal{S}a_{n_k}) = \rho(c, \mathcal{S}a) = \rho(\mathcal{A}, \mathcal{B}).$$

On account of the fact that $\mathcal{S}$ is proximally quasi-continuous (by (iv)), $\{a_{n_k}\}$ has a convergent subsequence to c. Thus, $c = a$ and we get $\rho(a, \mathcal{S}a) = \rho(\mathcal{A}, \mathcal{B})$. ■

Theorem 2.4 Let $(\mathcal{M}, \rho)$ be a complete metric space. Suppose that the following conditions are satisfied:
(i) $\mathcal{A}, \mathcal{B}$ are nonempty closed subsets of $\mathcal{M}$;
(ii) $\mathcal{A}_0$ is nonempty;
(iii) $\mathcal{S} : \mathcal{A} \to \mathcal{B}$ is a strong proximal $\mathcal{F}^*-$weak contraction of the first and second kind;
(v) $\mathcal{S}(\mathcal{A}_0) \subseteq \mathcal{B}_0$;
(vi) $\mathcal{F}$ is continuous.
Then there exists a unique element $a \in A$ such that $\rho(a, \mathcal{S}xa) = \rho(\mathcal{A}, \mathcal{B})$. Further, for any fixed $a_0 \in \mathcal{A}_0$, the sequence $\{a_n\}$ defined by $\rho(a_{n+1}, \mathcal{S}a_n) = \rho(\mathcal{A}, \mathcal{B})$ is convergent to a.

Proof. Let $a_0 \in \mathcal{A}_1$. Using Lemma 2.1 we get a sequence $\{a_n\}$ in $\mathcal{A}_0$ such that

$$\rho(a_{n+1}, \mathcal{S}a_n) = \rho(a_{n+2}, \mathcal{S}a_{n+1}) = \rho(\mathcal{A}, \mathcal{B}).$$

Similar to the proof of Theorem 2.2, it can be shown that, $\{a_n\} \subseteq \mathcal{A}$ is a Cauchy sequence. since $\mathcal{M}$ is complete and $\mathcal{A}$ is closed, then there exists $a \in \mathcal{A}$ such that $a_n \to a$ as $n \to +\infty$. Also similar to the proof of Theorem 2.2, $\mathcal{S}a_n$ is a Cauchy sequence. So there exists $b \in B$ such that $\mathcal{S}a_n \to b$ as $n \to +\infty$. Then $\rho(a_{n+1}, \mathcal{S}a_n) \to \rho(a, b) = \rho(\mathcal{A}, \mathcal{B})$ as $n \to +\infty$. Thus $a \in A_0$ and so there exists $c \in B_0$ such that

$$\rho(c, \mathcal{S}a) = \rho(a_{n+1}, \mathcal{S}a_n) = \rho(\mathcal{A}, \mathcal{B}).$$

Similar to the proof of Theorem 2.2 we can show that $c = a$ and a is the unique best proximity point of $\mathcal{S}$. ■

In next theorems we replace approximatively compact condition with others conditions.

Theorem 2.5 Let $(\mathcal{M}, \rho)$ be a metric space. Suppose that the following conditions are satisfied
(i) $\mathcal{A}, \mathcal{B}$ are nonempty closed subsets of $\mathcal{M}$;
(ii) $\mathcal{A}_0, \mathcal{B}_0$ are nonempty;
(iii) $(\mathcal{A}, \mathcal{B})$ is a fairly complete space;
(iv) $\mathcal{A}$ has uniform $\mathcal{S}-$approximation in $\mathcal{B}$;

(v) $\mathcal{S}:\mathcal{A}\to\mathcal{B}$ is a proximal $\mathcal{F}^*-$weak contraction of the first kind;
(vi) $\mathcal{S}(\mathcal{A}_0)\subseteq\mathcal{B}_0$;
(vii) $\mathcal{F}$ is continuous.
Then there exists $a\in\mathcal{A}$ such that $\rho(a,\mathcal{S}a)=\rho(\mathcal{A},\mathcal{B})$. Further, for any fixed $a_0\in\mathcal{A}_0$, the sequence $\{a_n\}$ defined by $\rho(a_{n+1},\mathcal{S}a_n)=\rho(\mathcal{A},\mathcal{B})$ is convergent to a.

Proof. Let $a_0\in\mathcal{A}_0$. Using Lemma 2.1, we get sequences $\{a_n\}$ in $\mathcal{A}_0$ and $\{b_n\}$ in $\mathcal{B}_0$ $(b_n=\mathcal{S}a_n)$ such that for any $n\in\mathbb{N}$, we have

$$\rho(a_{n+1},b_n)=\rho(a_{n+1},\mathcal{S}a_n)=\rho(\mathcal{A},\mathcal{B})$$

and

$$\rho(a_{n+1},\mathcal{S}a_n)=\rho(a_{n+2},\mathcal{S}a_{n+1})=\rho(\mathcal{A},\mathcal{B}).$$

Thus, by (v), we get

$$\tau+\mathcal{F}(\rho(a_{n+2},a_{n+1}))\leqslant\mathcal{F}(\rho(a_{n+1},a_n)). \tag{11}$$

for any $n\in\mathbb{N}$.

Similar to the proof of Theorem 2.2, it can be shown that $\{a_n\}\subseteq\mathcal{A}$ is a Cauchy sequence. Then by (iv), there exists $\delta>0$ such that for enough large m,n,

$$\rho(a_{n+1},b_n)=\rho(a_{m+1},b_m)=\rho(\mathcal{A},\mathcal{B}),\rho(a_{n+1},a_{m+1})<\delta\Longrightarrow\rho(b_n,b_m)<\epsilon.$$

for given $\epsilon>0$. Then $\{b_n\}$ is a Cauchy sequence. Also, we have

$$\rho(b_n,a_n)\leqslant\rho(b_n,a_{n+1})+\rho(a_{n+1},a_n)\leqslant\rho(\mathcal{A},\mathcal{B})+\rho(a_{n+1},a_n).$$

On the other hand,

$$\rho(a_m,b_n)\leqslant\rho(a_m,a_n)+\rho(a_n,b_n).$$

Therefore, (a_n,b_n) is a cyclically Cauchy sequence. By (iii), there exists $a\in A$ and $b\in B$ such that $a_n\to a$ and $b_n\to b$ as $n\to+\infty$. Thus, we have

$$\rho(a,b)=\lim_{n\to+\infty}\rho(a_{n+1},b_n)=\rho(\mathcal{A},\mathcal{B}).$$

Then for $a\in\mathcal{A}_0$ and by (vi), there exists $c\in\mathcal{A}$ such that $\rho(c,\mathcal{S}a)=\rho(\mathcal{A},\mathcal{B})$. Let $c\neq a$. For any $n\in\mathbb{N}$, we have $\rho(a_{n+1},\mathcal{S}a_n)=\rho(\mathcal{A},\mathcal{B})$. By (v), we can deduce that

$$\tau+\mathcal{F}(\rho(a_{n+1},c))\leqslant\mathcal{F}(\rho(a_n,c)).$$

That is a contradiction. Then $c=a$. Also Similar to the proof of Theorem 2.2 we can show that the best proximity point of $\mathcal{S}$ is unique and the proof is complete. ■

Theorem 2.6 Let $(\mathcal{M},\rho)$ to be a metric space. Suppose that the following conditions are satisfied:
(i) $\mathcal{A},\mathcal{B}$ are nonempty closed subsets of $\mathcal{M}$;
(ii) A_0,B_0 are nonempty;

(iii) (A, B) is a fairly complete space;
(iv) $\mathcal{BA}$ has a uniform approximation in $\mathcal{A}$;
(v) $\mathcal{S} : \mathcal{A} \to \mathcal{B}$ is a strong proximal $\mathcal{F}^*-$weak contraction of the second kind;
(vi) $\mathcal{S}(\mathcal{A}_0) \subseteq \mathcal{B}_0$;
(vii) $\mathcal{F}$ is continuous.
Then there exists a unique element $a \in \mathcal{A}$ such that $\rho(a, \mathcal{S}a) = \rho(\mathcal{A}, \mathcal{B})$. Further, for any fixed $a_0 \in \mathcal{A}_0$, the sequence $\{a_n\}$ defined by $\rho(a_{n+1}, \mathcal{S}a_n) = \rho(\mathcal{A}, \mathcal{B})$ is convergent to a.

Proof. Let $a_0 \in \mathcal{A}_1$. By using the proof of lemma 2.1, we get sequences $\{a_n\}$ in $\mathcal{A}_0$ and $\{b_n\}$ in $\mathcal{B}_0$ ($b_n = \mathcal{S}a_n$) such that

$$\rho(a_{n+1}, b_n) = \rho(a_{n+1}, \mathcal{S}a_n) = \rho(\mathcal{A}, \mathcal{B})$$

and

$$\rho(a_{n+1}, \mathcal{S}a_n) = \rho(a_{n+2}, \mathcal{S}a_{n+1}) = \rho(\mathcal{A}, \mathcal{B}).$$

for any $n \in \mathbb{N}$. Thus by (v) , for any $n \in \mathbb{N}$, we get

$$\tau + \mathcal{F}(\rho(\mathcal{S}a_{n+2}, \mathcal{S}a_{n+1})) \leqslant \mathcal{F}(\rho(\mathcal{S}a_{n+1}, \mathcal{S}a_n)). \tag{12}$$

Similar to the proof of Theorem 2.3, it can be shown that, $\{b_n = \mathcal{S}a_n\} \subseteq \mathcal{B}$ is a Cauchy sequence. Then by (iv), for given $\epsilon > 0$, there exists $\delta > 0$ such that for enough large m, n,

$$\rho(a_{n+1}, b_n) = \rho(a_{m+1}, b_m) = \rho(\mathcal{A}, \mathcal{B}), \rho(b_n, b_m) < \delta$$

which implies that $\rho(a_{n+1}, a_{m+1}) < \epsilon$. Then $\{a_n\}$ is a Cauchy sequence. Also, we have

$$\rho(b_n, a_n) \leqslant \rho(b_n, a_{n+1}) + \rho(a_{n+1}, a_n) \leqslant \rho(\mathcal{A}, \mathcal{B}) + \rho(a_{n+1}, a_n).$$

On the other hand, we have

$$\rho(a_m, b_n) \leqslant \rho(a_m, a_n) + \rho(a_n, b_n).$$

Therefore, (a_n, b_n) is a cyclically Cauchy sequence. By (iii), there exist $a \in A$ and $b \in B$ such that $a_n \to a$ and $b_n \to b$ as $n \to +\infty$. Thus we have

$$\rho(a, b) = \lim_{n \to +\infty} \rho(a_{n+1}, b_n) = \rho(\mathcal{A}, \mathcal{B}).$$

Then $a \in A_0$. By (vi), there exists $c \in A$ such that $\rho(c, \mathcal{S}a) = \rho(\mathcal{A}, \mathcal{B})$. Let $c \neq a$. For any $n \in \mathbb{N}$ we have $\rho(a_{n+1}, \mathcal{S}a_n) = \rho(\mathcal{A}, \mathcal{B})$. By (v) and that $\mathcal{S}$ is proximally quasi-continuous and $a_n \to a$, Then there exists subsequence $\{a_{n_r+1}\}$ of $\{a_{n+1}\}$ such that $a_{n_r+1} \to c$ as $r \to +\infty$. Thus we get $c = a$. Also Similar to the proof of Theorem 2.2 we can show that the best proximity point of $\mathcal{S}$ is unique and the proof is complete. ■

***Example* 2.7** Suppose $\mathcal{M} = \mathbb{R}^2$ endowed with the metric $\rho((u, v), (a, b)) = |u-a|+|v-b|$. Set $\mathcal{A} = \{(0, u) : 0 \leqslant u \leqslant n\}, B = \{(1, v) : 0 \leqslant v \leqslant n\}$ where $n \in \mathbb{N}$. It is clear that $\mathcal{A}_0 = \mathcal{A}, \mathcal{B}_0 = \mathcal{B}$, and $\mathcal{S}(\mathcal{A}_0) \subseteq \mathcal{B}_0$. Moreover, $\mathcal{B}$ is approximatively compact with respect to $\mathcal{A}$. Let us define $\mathcal{S}:\mathcal{A}\longrightarrow\mathcal{B}$ by $\mathcal{S}(0,u)=(1,\frac{u}{2})$. We show that $\mathcal{S}$ is a proximal $\mathcal{F}^*$-weak contraction of the first kind. Assume that

$$\rho(u_1,Tu)=\rho(u_2,Tv)=\rho(\mathcal{A},\mathcal{B})=1,$$

where $u_1,u_2,u,v\in A$. If we set $\mathcal{S}u=(1,\frac{u}{2}),\mathcal{S}v=(1,\frac{v}{2})$, then we get $u_1=(0,\frac{u}{2}),u_2=(0,\frac{v}{2})$. For $\tau=ln(2)$ and $\mathcal{F}(\alpha)=\ln(\alpha)+\sin(\frac{\alpha\pi}{2n})$, we have

$$\tau+\mathcal{F}(\rho(u_1,u_2))\leqslant\mathcal{F}(\rho(u,v))$$
$$\Updownarrow$$
$$\tau+\ln(\frac{1}{2}|u-v|)+\sin(\frac{|u-v|\pi}{4n})\leqslant\ln(|u-v|)+\sin(\frac{|u-v|\pi}{2n})$$
$$\Updownarrow$$
$$\tau+\ln(\frac{1}{2})\leqslant 0.$$

Therefore, $\mathcal{S}:\mathcal{A}\to\mathcal{B}$ is a proximal $\mathcal{F}^*-$weak contraction of the first kind. Then, by Theorem 2.2, there exists $u^*\in\mathcal{A}$ such that $\rho(u^*,\mathcal{S}u^*)=\rho(\mathcal{A},\mathcal{B})=1$. It is clear that $u^*=(0,0)$.

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