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Topological complexities of finite digital images

M. İs[a,*], İ. Karaca[a]

[a]*Department of Mathematics, Faculty of Science, Ege University, İzmir, Turkey.*

Abstract. Digital topological methods are often used in computing the topological complexity of digital images. We give new results on the relation between reducibility and digital contractibility in order to determine the topological complexity of a digitally connected finite digital image. We present all possible cases of the topological complexity TC of a finite digital image in $\mathbb{Z}$ and $\mathbb{Z}^2$. Finally, we determine the higher topological complexity TC_n of finite irreducible digital images independently of the number of points for $n > 1$.

1. Introduction

One of the main streams of topological robotics is to apply topological ideas to solve specific problems in engineering and computer science. On the other hand, digital topology has an important place in the studies of computer science. Topological robotics and digital topology have a common field of study and common methods. This suggests the following question: What results can one get in the subject of robotics by using topological methods on digital images? The answer gets inspired with the study of using discrete structures in computing topological complexity numbers.

Studies of topological robotics start by defining the notion of the topological complexity number of a path-connected topological space by Farber [15]. This number is an integer that indicates the complexity of the field where the robot moves. Many different methods, especially cohomology, are used in algebraic topology to determine the number

exactly (see [16] for a collection of the methods used). Contractibility of a topological space is so important if one wants to know the topological complexity number precisely. The topological complexity number of a contractible space is 1 and the converse is also true. If a topological complexity number of a topological space is 1, then the space must be contractible [15]. Rudyak improves the idea of this topological complexity definition and presents the higher topological complexity number of a topological space [26]. He proves that the special version of this new number corresponds to Farber's topological complexity number. Karaca and Is define the digital topological complexity number and the digital higher topological complexity number by moving the study to the field of digital topology [19, 20]. Digital topology is a discrete structure built on digital images at the point, so it assembles topological features without including a topology (see [2, 3, 5–13, 21, 23–25] for more information about digital topology, its applications, and digital geometry). This fundamental difference makes some of the topological methods useless in digital topology. For instance, the cohomological cup-product method is one of the well-known methods in usual topology to have a new bound for the topological complexity number [15]. But it does not work for digital images [19]. At this point, it is sometimes necessary to use new ways that comply with the rules of digital topology. It is not only a problem of studies of digital topological complexity but also a problem of studies in every aspect of digital topology. As an example, the Euler characteristic is not a homotopy invariant for digital images [14]. Staecker et al. have a new numerical homotopy invariant for digitally connected digital images and regard their invariant as 'true', which means that it is not an adaptation from topology [17]. They use the notions of reducibility and rigidity. In this paper, we examine a relation between digital contractibility and reducibility (partly rigidity). This leads us to have a characterization of finite digital images in $\mathbb{Z}$ and $\mathbb{Z}^2$ in terms of the topological complexity and the higher topological complexity.

The organization of the paper is as follows. In Section 2, we have a simple background of the digital setting and recall the definitions of the topological complexity and the higher topological complexity with some important properties. In Section 3, we start with proving that if X is an irreducible digital image, then the topological complexity of the image is greater than 1. We also demonstrate under what conditions the reducibility requires digital contractibility. We prove that if $X \subset \mathbb{Z}$ is a digitally connected finite image, then the topological complexity of the image is 1. After that, we examine the topological complexity of irreducible images having a finite number of points. Using this fact, we have the topological complexity number of all digitally connected finite digital images in $\mathbb{Z}^2$. We conclude that there is no digitally connected finite image in $\mathbb{Z}$ and $\mathbb{Z}^2$ such that the topological complexity of the image is greater than 2. In Section 4, we consider the diagonal map on a digital image X and define a new digital fibrational substitute of it. Then we find the digital higher topological complexity number of irreducible images with computing the digital Schwarz genus of the digital fibrational substitute. The topological complexity of the irreducible images is independent of the number of points. At the end of the paper, we state some open problems.

2.   Preliminaries

This section is planned to provide some backgrounds commonly used in digital topology and topological robotics.

A digital image is the basic element of the digital topology and consists of a set with a relation on this set. More precisely, (X, κ) is a digital image, where X is a finite

subset of $\mathbb{Z}^n$ for a positive integer n and κ is an adjacency relation on X [3]. On a digital image, unlike in topological spaces, there is an adjacency relation instead of topology and this relationship works as follows: Let X be a finite subset of $\mathbb{Z}^n$ and let $k \in \mathbb{Z}$ with $1 \leqslant k \leqslant n$. For any two distinct elements $x = (x_1, \cdots, x_n)$, $y = (y_1, \cdots, y_n) \in X$, x and y are called c_k−adjacent if we have $|x_i - y_i| = 1$ for at most k indices i, and $|x_j - y_j| \neq 1$ implies $x_j = y_j$ for all indices j [3, 22]. The notation $x \leftrightarrow_{c_k} y$ is used when x is adjacent to y. By this construction, we have $c_1 = 2$ adjacency in $\mathbb{Z}$, $c_1 = 4$ and $c_2 = 8$ adjacencies in $\mathbb{Z}^2$, and $c_1 = 6$, $c_2 = 18$ and $c_3 = 26$ adjacencies in $\mathbb{Z}^3$. Let (X, κ) and (Y, λ) be any digital images. Let (x_1, y_1) and (x_2, y_2) be any two points in the cartesian product image $X \times Y$. Then (x_1, y_1) and (x_2, y_2) are adjacent in $X \times Y$ if one of the following conditions holds [1]:

- $x_1 = x_2$ and $y_1 \leftrightarrow_\lambda y_2$; or
- $x_1 \leftrightarrow_\kappa x_2$ and $y_1 = y_2$; or
- $x_1 \leftrightarrow_\kappa x_2$ and $y_1 \leftrightarrow_\lambda y_2$.

Let (X, κ) be a digital image in $\mathbb{Z}^n$ and let p be any point in X. A κ−neighbor of p is the point that is κ−adjacent to p [18]. Let $(X, \kappa) \subset \mathbb{Z}^n$ be a digital image. X is called κ−connected if and only if for every pair of different points x, $y \in X$, there is a set $\{x_0, x_1, \cdots, x_m\}$ of points in X such that $x = x_0$, $y = x_m$ and $x_i \leftrightarrow_\kappa x_{i+1}$ for $i = 0, 1, \cdots, m - 1$ [18, 22]. Let $f : (X_1, \kappa_1) \to (X_2, \kappa_2)$ be a digital map such that $X_1 \subset \mathbb{Z}^{m_1}$ and $X_2 \subset \mathbb{Z}^{m_2}$. Then f is said to be (κ_1, κ_2)−continuous if $x \leftrightarrow_{\kappa_1} x'$ for any different points x, $x' \in X_1$, then either $f(x) = f(x')$ or $f(x) \leftrightarrow_{\kappa_2} f(x')$ in X_2 [3]. In addition, f is (κ_1, κ_2)−isomorphism if f is bijective, (κ_1, κ_2)−continuous and the inverse f^{-1} is (κ_2, κ_1)−continuous [5].

A set $[a, b]_{\mathbb{Z}} = \{z \in \mathbb{Z} : a \leqslant z \leqslant b\}$ is called a digital interval from a to b [4, 7]. Since the interval is a subset of $\mathbb{Z}$, it has 2−adjacency. If a digital map $f : [0, m]_{\mathbb{Z}} \to X$ is $(2, \kappa)$−continuous with $f(0) = x$ and $f(m) = y$, then f is a digital path from x to y in X [7]. The digital path f is a κ−loop if $f(0) = f(m)$. The product of two digital paths is defined in [21]: Let $f : [0, m]_{\mathbb{Z}} \to X$ and $g : [0, n]_{\mathbb{Z}} \to X$ be digital κ−paths with $f(m) = g(0)$. Then the product of f and g,

$$(f * g) : [0, m + n]_{\mathbb{Z}} \to X$$

is a $(2, \kappa)$−continuous function defined by

$$(f * g)(t) = \begin{cases} f(t), & t \in [0, m]_{\mathbb{Z}} \\ g(t - m), & t \in [m, m + n]. \end{cases}$$

Let (X, κ) and (Y, λ) be two digital images, and let $f, g : X \to Y$ be (κ, λ)−continuous maps. The maps f and g are (κ, λ)−homotopic if there exists $m \in \mathbb{Z}$ such that for all $x \in X$, there is a digital map $F : X \times [0, m]_{\mathbb{Z}} \to Y$ with $F(x, 0) = f(x)$ and $F(x, m) = g(x)$, for any fixed $t \in [0, m]_{\mathbb{Z}}$, the digital map $F_t : X \to Y$ is (κ, λ)−continuous, and for any fixed $x \in X$, the digital map $F_x : [0, m]_{\mathbb{Z}} \to Y$ is $(2, \lambda)$−continuous [3, 21]. It is denoted by $f \simeq_{(\kappa, \lambda)} g$ when f is (κ, λ)−homotopic to g. We also note that m is said to be the step number of the homotopy in this construction. In another saying, we say that f is digitally homotopic to g in m step.

Let $f : X \to Y$ be a (κ, λ)−continuous map. Then f is a (κ, λ)−homotopy equivalence if there exists a (λ, κ)−continuous map $g : Y \to X$ for which $g \circ f$ is digitally homotopic to the identity function on X and $f \circ g$ is digitally homotopic to the identity function

on Y [8]. A digital image X is said to be $\kappa-$contractible if the identity map on X is $(\kappa, \kappa)-$homotopic to a constant map at some $x_0 \in X$, i.e., the constant map $c : X \to X$ defined by $c(x) = x_0$ for all $x \in X$ [3, 21].

Definition 2.1 [17] Let (X, κ) be a finite digital image. If X is $(\kappa, \kappa)-$homotopy equivalent to an image of fewer points, then X is called reducible. If X is not reducible, then X is said to be irreducible.

Definition 2.2 [17] Let (X, κ) be a finite digital image. If the identity map on X is the only map that is $(\kappa, \kappa)-$homotopic to the identity map on X, then X is rigid.

Let (X, κ) be a digital image. We say that X is a digital simple closed $\kappa-$curve if there is an integer $m \geqslant 4$ for which there exists a $(2, \kappa)-$continuous map $f : [0, m - 1]_{\mathbb{Z}} \to X$ such that the following conditions hold [8, 22, 25]:

- f is bijective;
- $f(0) \leftrightarrow_\kappa f(m - 1)$; and
- for all $t \in [0, m-1]_{\mathbb{Z}}$, the only $\kappa-$neighbors of $f(t)$ in $f([0, m-1]_{\mathbb{Z}})$ are $f((t-1)mod\ m)$ and $f((t+1)mod\ m)$.

A simple closed curve with m points is generally denoted by C_m and named as an $m-$gon or a digital $m-$cycle. Let (X, κ) be a digital image. An $m-$loop is a digitally continuous map from C_m to X [17]. Moreover, the map $p : C_m \to X$ is called a simple $m-$loop if p is an injection with $p(c_i) \leftrightarrow_\kappa p(c_{i+1})$ in X such that there are no other adjacencies between points in the image of C_m.

Proposition 2.3 [17] C_m is irreducible for $m \geqslant 5$.

Definition 2.4 [17] $L_m(X)$ is an integer that counts the number of equivalence classes of $m-$loops for any finite digital image X.

Theorem 2.5 [17] Let (X, κ) and (Y, λ) be any two digital images such that they are digitally homotopy equivalent. Then for all positive integer m, we get $L_m(X) = L_m(Y)$.

The next three results are the basic facts that we often use in the next sections. By using these results, we have an idea about the digital topological complexity of a finite digital image (reducible or irreducible) with respect to the number of its points.

Proposition 2.6 [17] Let (X, κ) be a finite digital image. If X has no simple $m-$loop for any $m \geqslant 4$, then X is digitally homotopy equivalent to a one-point digital image $\{x\}$ for any $x \in X$, i.e., X is $\kappa-$contractible.

Proposition 2.7 [17] Let (X, κ) be a digitally connected digital image having m points. If $m \leqslant 4$, then X is digitally homotopy equivalent to a one-point digital image $\{x\}$ for any $x \in X$, i.e., X is $\kappa-$contractible.

Proposition 2.8 [17] Let X be a digitally connected digital image having five points. Then X is digitally homotopy equivalent to a one-point digital image or to C_5.

Let PX be a set of all digitally continuous digital paths for a $\kappa-$connected digital image (X, κ). Let $s : X \times X \to PX$ be the digital map which takes any pair (a, b) of a digital image to a digital path starting at a and ending at b, is denoted by the digital version of the motion planning algorithm. In [20], there is a reasoned way to define the continuity of motion planning algorithm. The digital connectedness on PX is defined as follows: Let τ be an adjacency relation on PX, and let α and β be any digital paths on X. If α and β are $\tau-$connected for all $t \in [0, m]_{\mathbb{Z}}$, then $\alpha \leftrightarrow_\kappa \beta$. α and β can have

different steps in their way. For instance, when α has 5 steps and β has 2 steps, the last step of β repeats itself 3 times. Then both α and β have the same number of steps, which means there is no confusion about the adjacency of digital paths. See [20] for more detail and example about the continuity of digital motion planning algorithms. Moreover, $\pi : PX \to X \times X$ is a digital map, which takes any digital path α to the pair $(\alpha(0), \alpha(m))$, where $\alpha(m)$ is the final step of α. Finally, we are ready to give the following definition:

Definition 2.9 [20] The digital topological complexity $\mathrm{TC}(X, \kappa)$ of a digital image (X, κ) is the minimum number k such that

$$X \times X = U_1 \cup U_2 \cup \cdots \cup U_k$$

with the property that there exists a digitally continuous motion planning algorithm $s_j : U_j \to PX$, $j = 1, 2, \cdots, k$, for which $\pi \circ s_j$ is identity map on each $U_j \subset X \times X$. If no such k exists, then we agree that $\mathrm{TC}(X, \kappa) = \infty$.

We compute the digital topological complexity of only connected digital images (recall that in ordinary topology, only path-connected topological spaces are considered for the computation of the topological complexity). The next proposition is quite important such as the fact that the topological complexity is a homotopy invariant.

Proposition 2.10 [20] $\mathrm{TC}(X, \kappa) = 1$ if and only if (X, κ) is $\kappa-$contractible.

Definition 2.11 [19] Let $f : (X, \kappa) \to (Y, \lambda)$ be a continuous map in digital images between digitally connected spaces (X, κ) and (Y, λ). A digital fibrational substitute of f is defined as a digital fibration $\widehat{f} : (Z, \kappa_3) \to (Y, \lambda)$ such that there exists a commutative diagram

$$
\begin{array}{ccc}
X & \xrightarrow{\ h\ } & Z \\
{\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \widehat{f}} \\
Y & \underset{1_Y}{=\!=\!=} & Y,
\end{array}
$$

where h is a digital homotopy equivalence.

Let $p : X \to Y$ be a digital fibration. The digital Schwarz genus of p is defined as the minimum number k such that $Y = U_1 \cup U_2 \cup \cdots \cup U_k$ with the property that for all $1 \leqslant i \leqslant k$, there is a digitally continuous map $s_i : U_i \to X$ with $p \circ s_i = id_{U_i}$ [19]. If we do not have a digital fibration, then we regard the digital Schwarz genus of a map as the digital Schwarz genus of its digital fibrational substitute. Consequently, we now give another important definition:

Definition 2.12 [19] Let X be any κ-connected digital image. Let J_n be the wedge of $n-$digital intervals $[0, m_1]_{\mathbb{Z}}, \cdots, [0, m_n]_{\mathbb{Z}}$ for a positive integer n, where $0_i \in [0, m_i]$, $i = 1, \cdots, n$, are identified. Then the digital higher topological complexity $\mathrm{TC}_n(X, \kappa)$ is defined by the digital Schwarz genus of the digital fibration

$$e_n : X^{J_n} \to X^n$$

$$f \longmapsto (f_1(m_1), \cdots, f_n(m_n)),$$

where $f = (f_1, \cdots, f_n)$ is a multipath on X.

In the definition of the higher topological complexity in digital images, we have $\text{TC}_2 = \text{TC}$ [19]. Furthermore, TC_n is also a homotopy invariant for digital images just as TC.

3. Digital topological complexity in $\mathbb{Z}$ and $\mathbb{Z}^2$

We begin with discussing the relation between the contractibility and the reducibility on digitally connected digital images. It is clear that if (X, κ) is a $\kappa-$contractible finite digital image, then X is reducible. The converse need not be true. For example, consider the following digital image $X \subseteq \mathbb{Z}^2$ with $8-$adjacency (see Figure 1(a)) and its digital homotopy equivalent digital image with $8-$adjacency (see Figure 1(b)). The
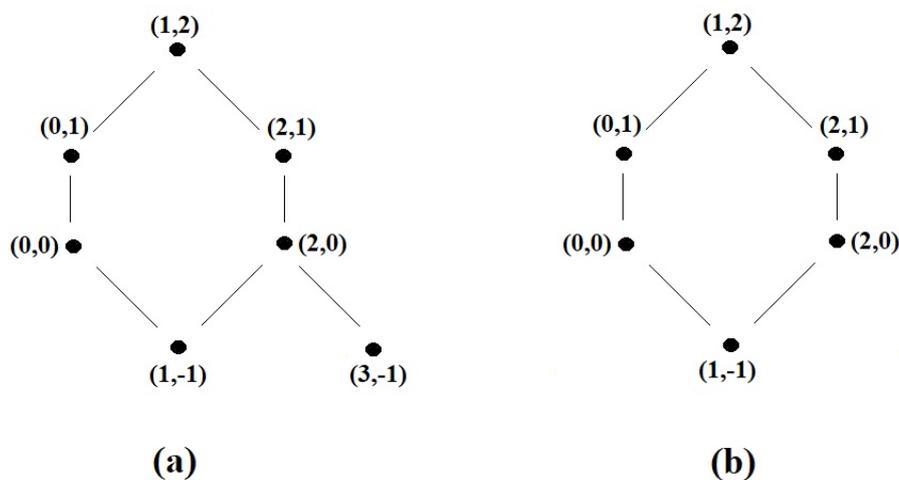

Figure 1. The digital image X with $8-$adjacency is on the left (a) and its digital homotopy equivalence $X \setminus \{(3, -1)\}$ on the right (b).

digital image X is reducible because it is digitally homotopy equivalent to the image $X \setminus \{(3, -1)\}$ (Figure 1 (b)) but it is well-known that X is not $8-$contractible. Combining this result with Proposition 2.10, we have that the topological complexity number of a reducible image can be different from 1. Indeed, we obtain that

$$\text{TC}(X, 8) = \text{TC}(X \setminus \{(3, -1)\}, 8) = 2$$

(see [Example 3.5, [20]]). In addition, if for any digitally connected finite image (X, κ) having more than one point, $\text{TC}(X, \kappa) = 1$, then X must be reducible. So, we immediately have the result:

Proposition 3.1 Let (X, κ) be a digitally connected finite image having more than one point. If X is irreducible, then $\text{TC}(X, \kappa) > 1$.

We note that Proposition 3.1 is still true if we choose X as a rigid digital image instead of an irreducible digital image. We express that the digital contractibility implies the reducibility. The next Lemma shows that the converse of this statement is also valid under certain requirements.

Lemma 3.2 Let X be a digital image with $m \geqslant 4$ points.

a) If no digital simple closed curve exists in X, then X is digitally contractible if and only if X is reducible.

b) If there is a digital simple closed curve in X, and X is not digitally homotopy equivalent to C_m, then X is digitally contractible if and only if X is reducible.

Proof. a) It is enough to prove that if X is reducible then X is digitally contractible. Let X be a reducible digital image. Then X is digitally homotopy equivalent to an image $X \setminus A$, where A has fewer points than X. Let $*$ be any point of X. If X is digitally homotopy equivalent to the one-point image $\{*\}$, then there is nothing to prove. Assume that X is not digitally homotopy equivalent to the one-point image. By Proposition 2.6, we have that X has a simple $m-$loop for any $m \geqslant 4$. Therefore, there exists a digitally continuous injection $p : C_m \to X$. This is a contradiction because p cannot be an injection. Whereas X has m points, there does not exist a digital simple closed curve in X for any $m \geqslant 4$. As a conclusion, X is digitally contractible.

b) Let X be a reducible digital image. Assume that X is not digitally homotopy equivalent to the one point image. Then we have a digitally continuous injection $p : C_m \to X$. The cardinality of C_m is m, and the cardinality of X is the same. This implies that p is surjective. Therefore, p is a bijection. If we define $q : X \to C_m$ with $q(x) = p^{-1}(x)$, then q is digitally continuous. Indeed, for any $x_i \in X$, $i = 1, \cdots, m$, we find $p^{-1}(x_i) \leftrightarrow p^{-1}(x_{i+1})$ because $p^{-1}(x_i) = c_i$ and $p^{-1}(x_{i+1}) = c_{i+1}$. Hence, we get $p \circ q = id_X$ and $q \circ p = id_X$. This means that X is digitally homotopy equivalent to C_m which is a contradiction. Finally, X is digitally homotopy equivalent to the one-point image, i.e., X is digitally contractible. ■

Lemma 3.3 A digitally connected image $X \subset \mathbb{Z}$ is $2-$contractible if and only if $L_1(X) = 1$.

Proof. Let $X \subset \mathbb{Z}$ be a $2-$contractible image. Then X is digitally homotopy equivalent to the one-point digital image $\{*\}$. We observe that the one-point is the unique irreducible image in $\mathbb{Z}$. By Theorem 2.5, $L_1(X) = L_1(\{*\}) = 1$. Conversely, if $L_1(X) = 1$, then we have that the number of equivalence classes of $1-$loops is 1. This means that X is $2-$contractible. ■

From the digital image X in Figure 1 (a), we cannot generalize Lemma 3.3 in $\mathbb{Z}^n$ for $n > 1$. Since X is $8-$connected, $L_1(X) = 1$. However, X is not $8-$contractible. The following Corollary is a result of Lemma 3.3 and Proposition 2.10.

Corollary 3.4 Let $X \subset \mathbb{Z}$ be a digitally connected finite image. Then $\mathrm{TC}(X, 2) = 1$.

We now provide the digital topological complexity numbers of digital simple closed curves in $\mathbb{Z}^2$.

Theorem 3.5 If there is a $\kappa-$connected digital simple closed curve in $\mathbb{Z}^2$ for any positive integer m, where $\kappa \in \{4, 8\}$, then

$$\mathrm{TC}(C_m, \kappa) = \begin{cases} 1, & m < 5 \\ 2, & m > 5. \end{cases}$$

Proof. There are two adjacency relations 4 and 8 in $\mathbb{Z}^2$ so we have two cases. First, consider the $4-$adjacency on C_m. We catalog the first 12 simple closed curves with respect to the number m in this case (see Figure 2). We note that some pictures can be different (but homotopy equivalent) in Figure 2. For instance, the points of C_2 can be drawn vertically. This does not affect the result as the digital topological complexity
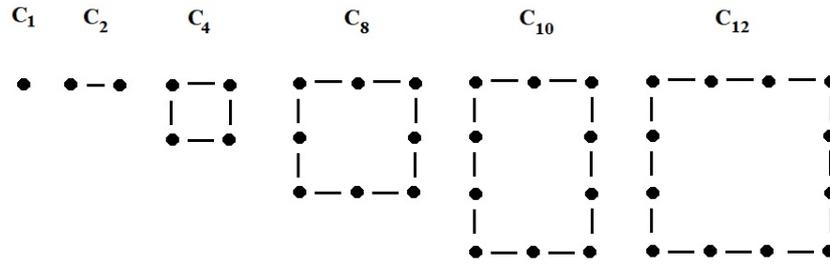
C_1 C_2 C_4 C_8 C_{10} C_{12}

Figure 2. Simple closed curve C_m related to $4-$adjacency for $m \in \{1, 2, 4, 8, 10, 12\}$.

number is a homotopy invariant for digital images. For $m > 12$, the list is extended. However, the computation of TC changes only when $m > 5$. Let $m < 5$. We have $\mathrm{TC}(C_m, 4) = 1$ because they are $4-$contractible digital images. If $m > 5$, then we show that $\mathrm{TC}(C_m, 4) = 2$. Let us choose any two diagonal points (the diagonal can be from left to right or from right to left) on any squares or rectangles for any $m > 5$ and split the graphic into two parts named U_1 and U_2. Without loss of generality, we assume that U_1 has one of the diagonal points and U_2 has the other point. Then U_1 and U_2 have the same number of points. We set

$$V_1 = \{(x, y) \in C_m \times C_m \mid (x, y) \in U_1\}$$

and

$$V_2 = \{(x, y) \in C_m \times C_m \mid (x, y) \in U_2 \text{ or } x \in U_1, y \in U_2 \text{ or } x \in U_2, y \in U_1\}$$

as the subsets of $C_m \times C_m$. Therefore, we get $C_m \times C_m = V_1 \cup V_2$. In addition, there exist digitally continuous sections $s_1 : V_1 \to PC_m$ and $s_2 : V_2 \to PC_m$ of a digital fibration $\pi : PC_m \to C_m \times C_m$. These satisfy that $\pi \circ s_1 = id_{V_1}$ and $\pi \circ s_2 = id_{V_2}$ and give the desired result for $4-$adjacency. Similarly, we list the first 8 simple closed curves with $8-$adjacency in Figure 3. For $m < 5$, C_m is $8-$contractible. Then we have that
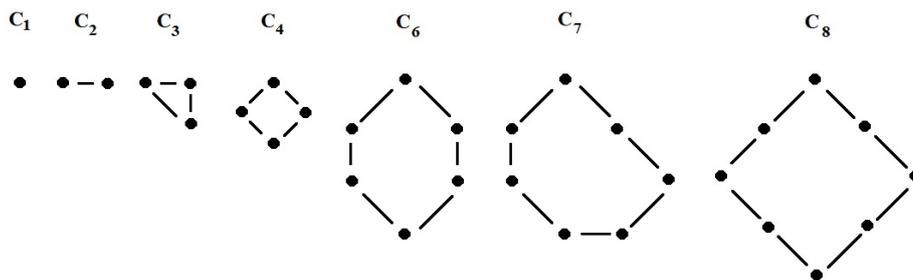
C_1 C_2 C_3 C_4 C_6 C_7 C_8

Figure 3. Simple closed curve C_m related to $8-$adjacency for $m \in \{1, 2, 3, 4, 6, 7, 8\}$.

$\mathrm{TC}(C_m, 8) = 1$. For $m > 5$, we choose the top and the bottom point of C_m (if there are one more top or bottom points, then choose one pair of them such that they are located vertically according to the each other) and split the graphic into two parts named as T_1 and T_2. Without loss of generality, we assume that T_1 has the bottom point and T_2 has

the top point. We set

$$W_1 = \{(x,y) \in C_m \times C_m \mid (x,y) \in T_1\}$$

and

$$W_2 = \{(x,y) \in C_m \times C_m \mid (x,y) \in T_2 \text{ or } x \in T_1, y \in T_2 \text{ or } x \in T_2, y \in T_1\}$$

as the subsets of $C_m \times C_m$. Then we have two digitally continuous sections $t_1 : W_1 \to PC_m$ and $t_2 : W_2 \to PC_m$ of a digital map $\pi : PC_m \to C_m \times C_m$ that satisfy that the digital maps $\pi \circ t_1$ and $\pi \circ t_2$ are equal to the identity maps. Moreover, there is no any digital simple closed curve with 5 points exists in X for both 4 and 8 adjacencies. This completes the proof. ∎

Corollary 3.6 Let $X \subset \mathbb{Z}^2$ be a digitally connected digital image with m points. $\mathrm{TC}(X,8) = 1$ for $m < 6$ and $\mathrm{TC}(X,4) = 1$ for $m < 8$.

Proof. Let $m \leqslant 4$. By Proposition 2.7, X is digitally homotopy equivalent to the one-point digital image. Then, we have that $\mathrm{TC}(X,\kappa) = 1$, where $\kappa \in \{4,8\}$. Let $m = 5$. From Proposition 2.8 and Proposition 2.10, we get $\mathrm{TC}(X,\kappa) = 1$, where $\kappa \in \{4,8\}$. Let $m = 6$ or 7. Then there does not exist any digital simple closed curve in X with respect to $4-$adjacency. So, X is digitally contractible because X is reducible. This shows that $\mathrm{TC}(X,4) = 1$ for $m = 6$ or $m = 7$. ∎

We are now ready to compute the topological complexity number of finite digital images in $\mathbb{Z}^2$. This characterization indicates that there are no finite digital images in $\mathbb{Z}^2$ whose topological complexity number is greater than 2.

Corollary 3.7 Let $(X,\kappa) \subset \mathbb{Z}^2$ be a $\kappa-$connected digital image with m points. If there is at least one digital simple closed curve exists in X, and X is digitally $(\kappa,\kappa)-$homotopy equivalent to C_m, then we get that

$$\mathrm{TC}(X,\kappa) = \begin{cases} 1, & \kappa = 4 \text{ and } m < 8 \\ 1, & \kappa = 8 \text{ and } m < 6 \end{cases}$$

and

$$\mathrm{TC}(X,\kappa) = \begin{cases} 2, & \kappa = 4 \text{ and } m \geqslant 8 \\ 2, & \kappa = 8 \text{ and } m \geqslant 6. \end{cases}$$

Proof. By Theorem 3.5 and Corollary 3.6, it is enough to show that

$$\mathrm{TC}(X,4) = \mathrm{TC}(X,8) = 1$$

when there is no digital simple closed curve exists in X or X is not digitally homotopy equivalent to C_m. Assume that there is a digital simple closed curve in X. If $m \leqslant 4$, then the result holds from Proposition 2.7. If $m \geqslant 5$, then we have that X is reducible from Proposition 2.3. Hence, the first part a) of Lemma 3.2 gives the desired result. Assume that the digital image X is not digitally homotopy equivalent to C_m. Then $\mathrm{TC}(X,\kappa) \neq 2$. Assume that there is a digital simple closed curve in X. Let $m \geqslant 5$. Since C_m is irreducible

for $m \geqslant 5$, X is reducible. Thus, the second part b) of Lemma 3.2 completes the proof. ∎

4.   Digital Higher Topological Complexity of Finite 2D Digital Images

We aim to give a general characterization of the digital higher topological complexity computations of any finite digital image especially in $\mathbb{Z}^2$ in this section.

We begin with computing the digital higher topological complexity TC_n of any one-point digital image for $n \geqslant 1$. Consider $X = \{*\} \subset \mathbb{Z}$ with $2-$adjacency. Let $f \in X^{J_n}$ be a constant map at $*$. The digital fibration $e_n : X^{J_n} \to X^n$, defined by $e_n(f) = (*, *, \cdots, *)$, has a digitally continuous map $s : X^n \to X^{J_n}$ with $s(*, *, \cdots, *) = f$ such that $e_n \circ s = id$. This shows that $\mathrm{TC}_n(X) = 1$.

Theorem 4.1 Let $(X, 2)$ be a finite $2-$connected digital image in $\mathbb{Z}$ and $n \geqslant 1$ be an integer. Then $\mathrm{TC}_n(X, 2) = 1$.

Proof. If $(X, 2)$ is finite and $2-$connected in $\mathbb{Z}$, then it is easy to see that X is $2-$contractible. Hence, it is $2-$homotopy equivalent to the one-point digital image. The digital homotopy invariance of TC_n gives the desired result. ∎

The digital higher topological complexity computation of a one-point digital image is quite useful because a great majority of digital images in $\mathbb{Z}^2$ are digitally contractible (have the same homotopy type as the one-point image). We now examine the digital higher topological complexity of another type which is not homotopy equivalent to the one-point image.

Lemma 4.2 Let (X, κ) be a $\kappa-$connected digital image. Consider the set

$$S_n(X) = \{(f, p_1, \cdots, p_n) \mid p_i \in Im(f), f \text{ is a digital path in } X, i = 1, 2, \cdots, n\}$$

in $X^{[0,m]_{\mathbb{Z}}} \times X^n$. Then the digital map

$$e_n' : S_n(X) \longrightarrow X^n$$
$$(f, p_1, p_2, \cdots, p_n) \longmapsto (p_1, p_2, \cdots, p_n)$$

is a digital fibrational substitute of the diagonal map $d_n : X \to X^n$.

Remark 1 *Note that the adjacency relation on $S_n(X)$ is defined as follows: For all $(f, p_1, p_2, \cdots, p_n)$, $(g, q_1, q_2, \cdots, q_n) \in S_n(X)$, $(f, p_1, p_2, \cdots, p_n)$ is κ_*-adjacent to $(g, q_1, q_2, \cdots, q_n)$ if f is $\lambda-$adjacent to g and p_i is $\kappa-$adjacent to q_i for all $i = 1, 2, \cdots, n$, where κ_* is an adjacency relation on $X^{[0,m]_{\mathbb{Z}}} \times X^n$ and λ is an adjacency relation on digital paths in X.*

Proof. Let $d_n : X \to X^n$ be a diagonal map of X. Define the digital map $h : X \to S_n(X)$ by $h(x) = (\epsilon_x, x, x, \cdots, x)$, where ϵ_x is the digital constant path at x. Let $(f, p_1, p_2, \cdots, p_n) \in S_n(X)$. Then there exists $y \in X$ such that $f(0) = y$. Since X is $\kappa-$connected, there exists a digital path g from x to y in X, i.e., $g(0) = x$ and $g(1) = f(0) = y$. To show that h is a digital homotopy equivalence, we define $k : S_n(X) \to X$ with $k(f, p_1, p_2, \cdots, p_n) = f * g(0)$. It is easy to see that $h \circ k$ is digitally homotopic to identity map on $S_n(X)$ and $k \circ h$ is digitally homotopic to identity map on

X. Moreover, we find

$$e_n' \circ h(x) = e_n'(\epsilon_x, x, \cdots, x) = (x, x, \cdots, x) = d_n(x).$$

Consequently, e_n' is a digital fibrational substitute of d_n. ∎

Lemma 4.3 $\mathrm{TC}_3(C_6, 8) = 2$.

Proof. Let

$$X = C_6 = \{p_1 = (0,0), p_2 = (1,1), p_3 = (2,1), p_4 = (3,0),$$
$$p_5 = (2,-1), p_6 = (1,-1)\},$$

where $p_1 < p_2 < p_3 < p_4 < p_5 < p_6$ (see Figure 4). Let $e_3' : S_3(X) \to X^3$ be a digital fibration with $e_3'(f, p_i, p_j, p_k) = (p_i, p_j, p_k)$ for $i, j, k \in \{1, 2, 3, 4, 5, 6\}$. We split X^3 into two parts. A_1 consists of triples in C_6 such that the order of points never changes from left to right, i.e., $p_i \leqslant p_j \leqslant p_k$ or if $p_i > p_j$, then $p_i = 6$ and $p_j = 1$ (similarly if $p_j > p_k$, then $p_j = 6$ and $p_k = 1$). A_2 consists of elements of C_6 in which they do not belong to A_1, i.e., the order of points can change from left to right except using 6 and 1 consecutively. Let $(p_i, p_j, p_k) \in A_1$. Using these points, we set a route starting and ending at p_i and p_k, respectively. Then we have a digitally continuous map $s_1 : A_1 \to S_3(X)$ with $s_1(p_i, p_j, p_k) = (f, p_i, p_j, p_k)$, where f is the route (digital path from p_i to p_k). It is clear that $e_3' \circ s_1 = id_{S_3(X)}$. Similarly, we can construct $s_2 : A_2 \to S_3(X)$ with $s_2(p_i, p_j, p_k) = (f, p_i, p_j, p_k)$ on A_2. Hence, we find that $e_3' \circ s_2 = id_{S_3(X)}$. Moreover, we have that $X^3 = A_1 \cup A_2$. As a result, we have that the digital Schwarz genus of e_3', $\text{genus}_{(\kappa_*, \lambda_*)}(e_3')$, is equal to 2, where κ_* and λ_* are adjacency relations on $S_3(X)$ and X^3, respectively. ∎
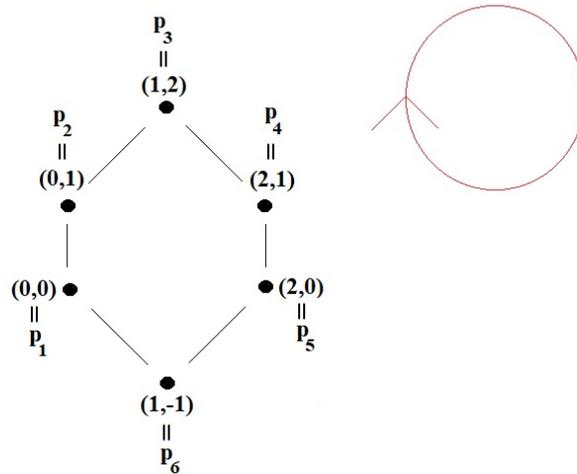

Figure 4. The irreducible digital image C_6 with the order of six points in it.

Lemma 4.4 $\mathrm{TC}_3(C_8, 4) = 2$.

Proof. Let

$$X = C_8 = \{r_1 = (0,0), r_2 = (0,1), r_3 = (0,2), r_4 = (1,2), r_5 = (2,2),$$
$$r_6 = (2,1), r_7 = (2,0), r_8 = (1,0)\},$$

where $r_1 < r_2 < r_3 < r_4 < r_5 < r_6 < r_7 < r_8$ (see Figure 5). In a similar way of Lemma 4.3, we get B_1 without changing the order of points and $t_1 : B_1 \rightarrow S_3(X)$ is a digitally continuous map on C_1 such that $e_3' \circ t_1$ is identity on B_1. Changing the order of points in C_8, we set B_2 that consists of triples in $C_8 \times C_8 \times C_8$. The digitally continuous map $t_2 : B_2 \rightarrow S_3(X)$ gives us $e_3' \circ t_2$ is identity on $S_3(X)$. Hence, we split X^3 into two parts B_1 and B_2. This proves that $\mathrm{TC}_3(X,4)$ is 2. ∎
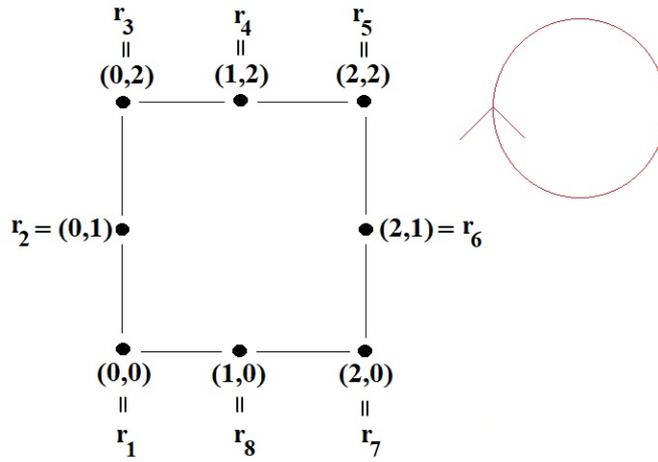

Figure 5. The irreducible digital image C_8 with the order of eight points in it.

Corollary 4.5 Assume that there is a digital $\kappa-$connected simple closed curve in X with m points. Then $\mathrm{TC}_3(C_m,4) = 2$ for $m \geqslant 8$ and $\mathrm{TC}_3(C_m,8) = 2$ for $m \geqslant 6$.

Proof. The proof is a generalization of Lemma 4.3 and Lemma 4.4. The order of points in C_m can be easily constructed for all cases. ∎

Corollary 4.5 can be improved for $n > 3$ and TC_n gives the same result with TC_3 for irreducible digital images:

Theorem 4.6 Assume that there is a digital simple closed curve in $\mathbb{Z}^2$ with m points. Let $n > 2$ be a positive integer. Then

- $\mathrm{TC}_n(C_m,4) = 2$ for $m \geqslant 8$,
- $\mathrm{TC}_n(C_m,8) = 2$ for $m \geqslant 6$.

Proof. Let $m \geqslant n$. Let $p_1, \cdots, p_m$ be points of C_m, where $p_1 < p_2 < \cdots < p_m$. By using the order, a digital path can be obtained by taking n or less (staying on the same point more than once) of m points. Then the method of Lemma 4.3 works for this case. Let $m < n$ and $(f, p_1, p_2, \cdots, p_n) \in S_n(X)$. In this case, it is necessary to increase the number of steps of the digital path to be able to have an $n-$step path created with m points. A new $n-$step path is obtained by adding the endpoint of any $m-$step path f to the end of the path $m - n$ times. Since we have $n-$step path, we use its n points in the definition of $S_n(X)$. After that, we split X^n into two parts A_1 and A_2 again: n points of

the digital image which follow the order and not, respectively. Thus, we conclude that the digital Schwarz genus of $e_n^{'}$ is 2. ∎

5. Conclusion

The aim of this paper is to characterize the digital topological complexity of digitally connected two-dimensional finite digital images entirely. We first deal with simple closed curves among digital images because they are irreducible. After giving the results about digital simple closed curves, we examine the topological complexity and the higher topological complexity of all possible digitally connected finite digital images in $\mathbb{Z}$ and $\mathbb{Z}^2$.

One of the open problems on this topic is to apply our work on $3-$dimensional digital images. As the number of points that a digital image has in three-dimensional space extremely increases, it is not easy to categorize the topological complexities of these points. Before solving this problem, it is more convenient that try to categorize the digital images up to digital homotopy equivalence, because of the fact that the topological complexity (and the higher topological complexity) is a homotopy invariant for digital images. Moreover, one can observe the results of the topological complexities of reducible or irreducible images in $\mathbb{Z}^3$. This leads us to think more about the characterize digital images up to the digital homotopy equivalence in any dimension of digital topology.

Acknowledgments

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