Journal of Linear and Topological Algebra
Vol. 11*, No.* 02*,* 2022*,* 77- 83
DOR: 20.1001.1.22520201.2022.11.02.1.4
DOI: 10.30495/JLTA.2022.1952840.1471

On the h-Jensen's operator inequality

S. S. Hashemi Karouei[a], M. S. Asgari[a,*], M. Shah Hosseini[b], N. Ghafoori Adl[a,*]

[a] *Department of Mathematics, Faculty of Science, Central Tehran Branch Islamic Azad University, Tehran, Iran.*
[b] *Department of Mathematics, Shahr-e-Qods Branch, Islamic Azad University, Tehran, Iran.*

Received 16 February 2022; Revised 9 April 2022; Accepted 14 April 2022.

Communicated by Hamidreza Rahimi

Abstract. In this paper, we prove Jensen's operator inequality for an h-convex function and we point out the results for classes of continuous fields of operators. Also, some generalizations of Jensen's operator inequality and some properties of the h-convex function are given.

Keywords: h-Convex function, h-concave function, h-Jensen's operator inequality.

2010 AMS Subject Classification: 26D15, 26D99.

1. Introduction

Inequalities play an important role in almost all branches of mathematics as well as in other areas of science. Convex functions have received considerable attention in the literature due to their applications in many scientific fields, such as mathematical inequalities, mathematical analysis, and mathematical physics. In the paper, [10] a large class of non-negative functions, the so-called h-convex functions are considered. This class contains several well-known classes of functions such as non-negative convex functions, s-convex in the second sense [4, 6, 11], Godunova-Levin functions [8], and P-functions [7]. For further details see [1–3, 5].

We fisrt recall here some concepts of h-convexity that are well known in the literature.

Definition 1.1 Let I, J be intervals in $\mathbb{R}$, $(0, 1) \subset J$, and let $h : J \to \mathbb{R}$ be a non-negative function, $h \neq 0$. A non-negative function $f : I \to \mathbb{R}$ is called h-convex if for all

---

*Corresponding author.
E-mail address: seyyed.s.hashemi@gmail.com (S. S. Hashemi Karouei); moh.asgari@iauctb.ac.ir, msasgari@yahoo.com (M. S. Asgari); mohsen_shahhosseini@yahoo.com (M. Shah Hosseini); naser.ghafoori@gmail.com (N. Ghafoori Adl).

Print ISSN: 2252-0201
Online ISSN: 2345-5934

http://jlta.iauctb.ac.ir

$x, y \in I$, $\lambda \in (0,1)$, we have

$$f(\lambda x + (1-\lambda)y) \leqslant h(\lambda)f(x) + h(1-\lambda)f(y). \tag{1}$$

If the inequality in (1) is reversed, then f is said to be h-concave.

Remark 1 *It follows from the above definition that if f is a non-zero h-convex (h-concave) function then $h(\lambda) + h(1-\lambda) \geqslant 1$ $\big(h(\lambda) + h(1-\lambda) \leqslant 1\big)$ for all $\lambda \in (0,1)$.*

Definition 1.2 A non-negative function $h : J \to \mathbb{R}$ is said to be a super-multiplicative function if

$$h(xy) \geqslant h(x)h(y) \tag{2}$$

for all $x, y \in J$. If inequality (2) is reversed, then h is said to be a submultiplicative function. If the equality holds in (2), then h is said to be a multiplicative function.

***Example* 1.3** Let $0 < a < b$ and $h \neq 0$ be a non-negative function such that $h(\lambda) \geqslant \lambda$ for all $\lambda \in (0,1)$. Then the function f defined by

$$f : I = [a,b] \to \mathbb{R}, \qquad f(x) = \begin{cases} a & x \neq \frac{a+b}{2} \\ b & x = \frac{a+b}{2} \end{cases}$$

is a non-convex function, but it is h-convex. In addition, if $h(\lambda) \leqslant \lambda$ for all $\lambda \in (0,1)$ then the function f defined by

$$f : I = [a,b] \to \mathbb{R}, \qquad f(x) = \begin{cases} b & x \neq \frac{a+b}{2} \\ a & x = \frac{a+b}{2} \end{cases}$$

is a non-concave function, but it is h-concave. Moreover, if $h \neq 0$ is a non-negative function such that $h(\lambda) \geqslant \lambda$ (resp., $h(\lambda) \leqslant \lambda$), then every convex (resp., concave) function is h-convex (resp., h-concave).

If h is a super-multiplicative or a submultiplicative function, then some very interesting results for h-convex functions are arised. For example, we give a simple characterization of h-convex functions, which is analogous to the characterization of convex functions.

Theorem 1.4 Let $h : [0,+\infty) \to \mathbb{R}$ be a super-multiplicative function. Let $f : I \to \mathbb{R}$ be a h-convex function. Then for $x, y, z \in I$, $x < y < z$ the following inequality holds:

$$h(z-x)f(y) \leqslant h(z-y)f(x) + h(y-x)f(z). \tag{3}$$

Moreover, if h is multiplicative then h-convexity f is equivalent to (3).

Proof. Let f be h-convex and $x, y, z \in I$, $x < y < z$. Then we have

$$\begin{aligned} f(y) = f\Big(\frac{z-y}{z-x}x + \frac{y-x}{z-x}z\Big) &\leqslant h\Big(\frac{z-y}{z-x}\Big)f(x) + h\Big(\frac{y-x}{z-x}\Big)f(z) \\ &\leqslant \frac{h(z-y)}{h(z-x)}f(x) + \frac{h(y-x)}{h(z-x)}f(z), \end{aligned}$$

which imply that (3). To prove the moreover part, suppose $x, y \in I$ with $x < y$ and $0 < \lambda < 1$. Then $x < \lambda x + (1-\lambda)y < y$ and by the assumptions we have

$$h(y-x)f\big(\lambda x + (1-\lambda)y\big) \leqslant h\big(y - \lambda x - (1-\lambda)y\big)f(x) + h\big(\lambda x + (1-\lambda)y - x\big)f(y)$$

and

$$h(y-x)f\big(\lambda x + (1-\lambda)y\big) \leqslant h\big(\lambda(y-x)\big)f(x) + h\big((1-\lambda)(y-x)\big)f(y).$$

Thus

$$h(y-x)f\big(\lambda x + (1-\lambda)y\big) \leqslant h(y-x)h(\lambda)f(x) + h(y-x)h(1-\lambda)f(y).$$

Dividing both sides of this inequality by $h(y-x)$ completes the proof. ■

Similarly, there is the following characterization of h-concave functions.

Corollary 1.5 Let $h : [0, +\infty) \to \mathbb{R}$ be a submultiplicative function and $f : I \to \mathbb{R}$ be a h-concave function. Then for $x, y, z \in I$, $x < y < z$ the following inequality holds:

$$h(z-x)f(y) \geqslant h(z-y)f(x) + h(y-x)f(z). \tag{4}$$

Moreover, if h is multiplicative then h-concavity f is equivalent to (4).

We begin with the following variant of Jensen's inequality which we call the h-Jensen's inequality.

Theorem 1.6 Suppose that I, J are intervals in $\mathbb{R}$, $(0,1) \subset J$, and $h : J \to \mathbb{R}$ is a non-negative super-multiplicative function, $h \neq 0$. Let $f : I \to \mathbb{R}$ be a h-convex function. Then for every $x_i \in I$ and all $t_i \geqslant 0$ with $\sum\limits_{i=1}^{n} t_i = 1$, we have

$$f\big(\sum_{i=1}^{n} t_i x_i\big) \leqslant \sum_{i=1}^{n} h(t_i)f(x_i), \quad n \in \mathbb{N},\ n \geqslant 2. \tag{5}$$

If h is submultiplicative and f is h-concave, then inequality (5) is reversed.

Proof. We proceed by induction on n. For $n = 2$ it is just (1). Now, if (5) holds for $k = n-1$, then given $t_i \geqslant 0$, $x_i \in I$ and $\sum\limits_{i=1}^{n} t_i = 1$. Define $s_i = \dfrac{t_i}{1-t_n}$ for all $i = 1, \cdots, n-1$. Then we have

$$\sum_{i=1}^{n} t_i x_i = (1-t_n)\big(\sum_{i=1}^{n-1} s_i x_i\big) + t_n x_n.$$

So applying the h-convexity of f, we have

$$f\big(\sum_{i=1}^{n} t_i x_i\big) \leqslant h(1-t_n)f\big(\sum_{i=1}^{n-1} s_i x_i\big) + h(t_n)f(x_n).$$

Now, by inductive hypothesis, we have

$$f\big(\sum_{i=1}^{n} t_i x_i\big) \leqslant h(1-t_n)\sum_{i=1}^{n-1} h(s_i)f(x_i) + h(t_n)f(x_n)$$
$$\leqslant h(1-t_n)\sum_{i=1}^{n-1} \frac{h(t_i)}{h(1-t_n)} f(x_i) + h(t_n)f(x_n) = \sum_{i=1}^{n} h(t_i)f(x_i).$$

■

Corollary 1.7 Let $h \neq 0$ be a non-negative super-multiplicative and let $f : I \to \mathbb{R}$ be a h-convex. Then

$$f\big(t^{-1}\sum_{i=1}^{n} t_i x_i\big) \leqslant (h(t))^{-1}\sum_{i=1}^{n} h(t_i)f(x_i) \tag{6}$$

for all $t_i \geqslant 0$, $x_i \in I$ and $0 \neq t = \sum_{i=1}^{n} t_i$. Moreover, if h is submultiplicative and f is h-concave, then inequality (6) is reversed.

Corollary 1.8 Let $h \neq 0$ be a non-negative super-multiplicative and let $f : I \to \mathbb{R}$ be a h-convex. Then

$$f\big(n^{-1}\sum_{i=1}^{n} x_i\big) \leqslant (h(n))^{-1}\sum_{i=1}^{n} f(x_i) \tag{7}$$

for all $n \in \mathbb{N}$, and $x_i \in I$. Moreover, if h is submultiplicative and f is h-concave, then inequality (7) is reversed.

2. h-Jensen's operator inequality

In what follows, $\mathcal{A}$ and $\mathcal{B}$ are C^*-algebras. We denote by $\mathcal{A}_h$ the real subspace of all self-adjoint elements in $\mathcal{A}$ and by $\mathcal{A}_+$ the set of all positive elements in $\mathcal{A}$. Here the symbol $1_\mathcal{A}$ denotes the identity operator in $\mathcal{A}$. We write $x \in \mathcal{A}_+ (x \geqslant 0)$ to mean that $x \in \mathcal{A}_h$ and $\mathrm{Sp}(x) \subset [0, +\infty)$. If $x - y \geqslant 0$, then we write $x \geqslant y$. A linear map $\Phi : \mathcal{A} \to \mathcal{B}$ is positive if $\Phi(x) \geqslant 0$ whenever $x \geqslant 0$. It is said to be unital if $\Phi(1_\mathcal{A}) = 1_\mathcal{B}$. For a real-valued function f of a real variable and a self-adjoint element $x \in \mathcal{A}_h$, the value $f(x)$ is understood by means of the functional calculus for self-adjoint operators.

Definition 2.1 A non-negative continuous function $f : I \to \mathbb{R}$ is said to be operator h-convex if

$$f(\lambda A + (1-\lambda)B) \leqslant h(\lambda)f(A) + h(1-\lambda)f(B) \tag{8}$$

holds for each $\lambda \in [0, 1]$ and every pair of self-adjoint operators A and B acting on an infinite dimensional Hilbert space $\mathcal{H}$ with spectra in I. If the inequality in (8) is reversed, then f is said to be operator h-concave.

The following example shows that if a function is convex, there is no guarantee that it is an operator h-convex.

***Example* 2.2** Let $h(t) = t^4$ on $[0, 1]$ and $f(t) = t^3$ on $[0, +\infty)$. Then the function f is convex but is not operator h-convex. To see this, let

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix}.$$

Then we have

$$\begin{aligned} h(\frac{1}{2})f(A) + h(\frac{1}{2})f(B) - f\Big(\frac{A+B}{2}\Big) &= \frac{1}{16}(A^3 + B^3) - \Big(\frac{A+B}{2}\Big)^3 \\ &= \frac{1}{8}\begin{pmatrix} 6 & 1 \\ 1 & 0 \end{pmatrix}, \end{aligned}$$

which is not positive.

The following theorem shows a generalization of Jensen's operator inequality for the h-convex functions.

Theorem 2.3 Let I be an interval such that $0 \in I$ and let $f : I \to \mathbb{R}$ be an operator h-convex function with $f(0) \leqslant 0$. Then $f(A^*XA) \leqslant 2h(\frac{1}{2})A^*f(X)A$ for all $A \in B(\mathcal{H})$ with $\|A\| \leqslant 1$ and all self-adjoint operator $X \in B(\mathcal{H})$ with spectrum in I.

Proof. Define the operators Y, U, V on $\mathcal{H} \oplus \mathcal{H}$ as follows:

$$Y = \begin{pmatrix} X & 0 \\ 0 & 0 \end{pmatrix}, \qquad U = \begin{pmatrix} A & B \\ C & -A^* \end{pmatrix}, \qquad V = \begin{pmatrix} A & -B \\ C & A^* \end{pmatrix},$$

where $B = \big(1 - AA^*\big)^{\frac{1}{2}}$ and $C = \big(1 - A^*A\big)^{\frac{1}{2}}$. Then U and V are unitaries and we obtain

$$U^*YU = \begin{pmatrix} A^*XA & A^*XB \\ BXA & BXB \end{pmatrix}, \qquad V^*YV = \begin{pmatrix} A^*XA & -A^*XB \\ -BXA & BXB \end{pmatrix}.$$

Now, from the operator h-convexity f we have

$$\begin{aligned} \begin{pmatrix} f(A^*XA) & 0 \\ 0 & f(BXB) \end{pmatrix} &= f\begin{pmatrix} A^*XA & 0 \\ 0 & BXB \end{pmatrix} = f(\frac{1}{2}U^*YU + \frac{1}{2}V^*YV) \\ &\leqslant h(\frac{1}{2})f(U^*YU) + h(\frac{1}{2})f(V^*YV) \\ &= h(\frac{1}{2})U^*f(Y)U + h(\frac{1}{2})V^*f(Y)V \\ &= h(\frac{1}{2})U^*\begin{pmatrix} f(X) & 0 \\ 0 & f(0) \end{pmatrix}U + h(\frac{1}{2})V^*\begin{pmatrix} f(X) & 0 \\ 0 & f(0) \end{pmatrix}V \\ &\leqslant h(\frac{1}{2})U^*\begin{pmatrix} f(X) & 0 \\ 0 & 0 \end{pmatrix}U + h(\frac{1}{2})V^*\begin{pmatrix} f(X) & 0 \\ 0 & 0 \end{pmatrix}V \\ &= \begin{pmatrix} 2h(\frac{1}{2})A^*f(X)A & 0 \\ 0 & 2h(\frac{1}{2})Bf(X)B \end{pmatrix}. \end{aligned}$$

In particular, we have $f(A^*XA) \leqslant 2h(\frac{1}{2})A^*f(X)A$. ■

Corollary 2.4 Suppose that all assumptions of Theorem 2.3 hold. Then

$$f(A^*XA+B^*YB) \leqslant 2h(\frac{1}{2})\big(A^*f(X)A+B^*f(Y)B\big)$$

for all $A,B \in B(\mathcal{H})$ with $A^*A+B^*B \leqslant 1$ and all self-adjoint operators $X,Y \in B(\mathcal{H})$ with spectrum in I.

Proof. Define the operators C,D on $\mathcal{H}\oplus\mathcal{H}$ as follows:

$$C=\begin{pmatrix} A & 0 \\ B & 0 \end{pmatrix} \quad \text{and} \quad D=\begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix}.$$

Then $\|C\| \leqslant 1$ and spectrum D is in I, so by Theoren 2.3 we obtain

$$\begin{aligned}
\begin{pmatrix} f(A^*XA+B^*YB) & 0 \\ 0 & f(0) \end{pmatrix} &= f\begin{pmatrix} A^*XA+B^*YB & 0 \\ 0 & 0 \end{pmatrix} \\
&= f(C^*DC) \leqslant 2h(\frac{1}{2})C^*f(D)C \\
&= 2h(\frac{1}{2})\begin{pmatrix} A^*f(X)A+B^*f(Y)B & 0 \\ 0 & 0 \end{pmatrix}.
\end{aligned}$$

In particular, we deduce the desired inequality. ■

Corollary 2.5 Suppose that all assumptions of Theorem 2.3 hold. Then

$$f(\sum_{i=1}^{N} A_i^*X_iA_i) \leqslant 2h(\frac{1}{2})\sum_{i=1}^{N} A_i^*f(X_i)A_i, \quad (N \in \mathbb{N})$$

for all $\{A_i\}_{i=1}^N \subset B(\mathcal{H})$ with $\sum\limits_{i=1}^{N} A_i^*A_i \leqslant 1$ and all self-adjoint operators $\{A_i\}_{i=1}^N \subset B(\mathcal{H})$ with spectrum in I.

Corollary 2.6 Suppose that all assumptions of Theorem 2.3 hold. Then $f(PXP) \leqslant 2h(\frac{1}{2})Pf(X)P$ for all projection $P \in B(\mathcal{H})$ and each self-adjoint operator $X \in B(\mathcal{H})$ with spectrum in I.

Corollary 2.7 Let $f: I \to \mathbb{R}$ be an operator h-convex function with $0 \in I$, $f(0) \leqslant 0$. Let $\mathcal{A}$ be a C^*-algebra, and $\pi: \mathcal{A} \to B(\mathcal{H})$ be a positive, linear contraction on $\mathcal{A}$. Then $f\big(\pi(x)\big) \leqslant 2h(\frac{1}{2})\pi\big(f(x)\big)$ for each self-adjoint operator $x \in \mathcal{A}$ with spectrum in I.

Proof. Restricting π to the commutative C^*-algebra generated by x. According to Stinespring's decomposition theorem [9], there is a $*$-homomorphism $\rho: \mathcal{A} \to B(\mathcal{K})$ on a Hilbert space $\mathcal{K}$ and an operator $A \in B(\mathcal{H},\mathcal{K})$ with $\|A\| \leqslant 1$ such that $\pi(x) = A^*\rho(x)A$ for all $x \in \mathcal{A}$. By extending Theorem 2.3 to this situation (for example by making a similar argument on $\mathcal{H}\oplus\mathcal{K}$) we obtain

$$f\big(\pi(x)\big) = f\big(A^*\rho(x)A\big) \leqslant 2h(\frac{1}{2})A^*f\big(\rho(x)\big)A = 2h(\frac{1}{2})A^*\rho\big(f(x)\big)A = 2h(\frac{1}{2})\pi\big(f(x)\big)$$

for all self-adjoint operator $x \in \mathcal{A}$. ■

Suppose that Ω is a locally compact Hausdorff space and μ is a positive Radon measure on Ω. Recall that a map $\Phi : \Omega \to \mathbb{B}(\mathcal{A}, \mathcal{B})$ is called a field of positive linear operators if $\Phi(t)(x) \geqslant 0$ whenever $x \geqslant 0$ and $t \in \Omega$, where $\mathbb{B}(\mathcal{A}, \mathcal{B})$ is the space of all bounded linear operators from $\mathcal{A}$ to another C^*-algebra $\mathcal{B}$. We say that such a field is continuous if the function $t \to \Phi(t)(x)$ is continuous for all $x \in \mathcal{A}$. If the C^*-algebras are unital and the function $t \to \Phi(t)(1_{\mathcal{A}})$ is integrable with integral $1_{\mathcal{B}}$, we say that Φ is unital.

Theorem 2.8 Let $f : I \to \mathbb{R}$ be an operator h-convex function with $0 \in I$, $f(0) \leqslant 0$. Let $\mathcal{A}$ and $\mathcal{B}$ be unital C^*-algebras. If $\Phi : \Omega \to \mathbb{B}(\mathcal{A}, \mathcal{B})$ is an unital field of positive linear operators defined on a locally compact Hausdorff space Ω with a bounded Radon measure μ, then the inequality

$$f\Big(\int_\Omega \Phi(t)(\chi(t))d\mu(t)\Big) \leqslant 2h(\frac{1}{2})\int_\Omega \Phi(t)(f(\chi(t)))d\mu(t)$$

holds for all bounded continuous field $\chi : \Omega \to \mathcal{A}$ of self-adjoint elements in $\mathcal{A}$ with spectra contained in I.

Proof. We first note that the function $t \to \Phi(t)(\chi(t))$ is continuous and bounded, hence integrable with respect to the bounded Radon measure μ. Let $C(\Omega, \mathcal{A})$ be the C^*-algebra of bounded functions on Ω with values in $\mathcal{A}$. A self-adjoint element $\chi \in C(\Omega, \mathcal{A})$ has spectra in I if for every $t \in \Omega$ the self-adjoint element $\chi(t) \in \mathcal{A}$ has spectra contained in I. In this case we write $f(\chi)(t) = f(\chi(t))$. Define the mapping

$$\pi : C(\Omega, \mathcal{A}) \to \mathcal{B}, \qquad \pi(\chi) = \int_\Omega \Phi(t)(\chi(t))d\mu(t), \qquad \forall \chi \in C(\Omega, \mathcal{A}).$$

Then π is an unital positive linear map and by corollary 2.7 we obtain

$$f\Big(\int_\Omega \Phi(t)(\chi(t))d\mu(t)\Big) = f(\pi(\chi)) \leqslant 2h(\frac{1}{2})\pi(f(\chi)) = 2h(\frac{1}{2})\int_\Omega \Phi(t)(f(\chi(t)))d\mu(t).$$

■

References

[1] A. O. Akdemira, M. E. Özdemirb, S. Varošanec, On some inequalities for h-concave functions, Math. Computer. Model. 55 (2012), 746-753.
[2] M. W. Alomari, A note on h-convex functions, e-J. Anal. Appl. Math. 2019 (1) (2019), 55-67.
[3] G. D. Anderson, M. K. Vamanamurthy, M. Vuorinen, Generalized convexity and inequalities, J. Math. Anal. Appl. 335 (2007), 1294-1308.
[4] W. W. Breckner, Stetigkeitsaussagen für eine Klasse verallgemeinerter konvexer funktionen in topologischen linearen Räumen, Publ. Inst. Math. 23 (1978), 13-20. (in German)
[5] S. S. Dragomir, Inequalities of Jensen type for h-convex functions on linear spaces, Mathematica Moravica. 19 (1) (2015), 107-121.
[6] S. S. Dragomir, S. Fitzpatrick, The Jensen inequality for s-Breckner convex functions in linear spaces, Demonstratio Mathematica. 33 (1) (2000), 43-49.
[7] S. S. Dragomir, J. Pečarić, L. E. Persson, Some inequalities of Hadamard type, Soochow J. Math. 21 (1995), 335-341.
[8] E. K. Godunova, V. I. Levin, Neravenstva dlja funkcii širokogo klassa, soderžaščego vypuklye, monotonnye i nekotorye drugie vidy funkcii, in: Vyčislitel. Mat. i. Mat. Fiz. Mežvuzov. Sb. Nauč. Trudov, MGPI, Moskva, 1985, 138-142.
[9] W. F. Stinespring, Positive functions on C^*-algebras, Proc. Amer. Math. Soc. 6 (2) (1955), 211-216.
[10] S. Varošanec, On h-convexity, J. Math. Anal. Appl. 326 (2007), 303-311.
[11] Gh. Zabandan, A. Bodaghi, A. Klman, The Hermite-Hadamard inequality for r-convex functions, J. Inequal. Appl. (2012), 2012:215.