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Topological spaces induced by homotopic distance

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Abstract. Topological complexity which plays an important role in motion planning problem can be generalized to homotopic distance D as introduced in [6]. In this paper, we study the homotopic distance and mention that it can be realized as a pseudometric on $\mathrm{Map}(X, Y)$. Moreover we study the topology induced by the pseudometric D. In particular, we consider the space $\mathrm{Map}(S^1, S^1)$ and use the non-compactness of it to talk about the non-compactness of $\mathrm{Map}(X, Y)$.

Keywords: Homotopic distance, metric spaces, topological complexity, Lusternik Schnirelmann category.

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1. Introduction and preliminaries

In [4], Michael Farber introduced a new concept in topology which is related to computer sciences and engineering based on the theory of robot motion planning by Latombe et al. [5]. Farber considered the following problem: Roughly speaking, one asks if it is possible to control a robot's motion from any point A to any point B in X using only one rule (continuous function). More precisely, one asks if it is possible to write only one continuous function which assings any pair $(A, B) \in X \times X$ a path in X starting at A and terminating at B. In [4], Farber showed that it is possible to find only one such a function if and only if X is contractible. For non-contractible spaces, Farber introduced a notion called topological complexity (denoted by TC). In some sense, topological complexity measures how far a space is away from admitting such a function.

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Homotopic distance which is introduced by Macías-Virgós and Mosquera-Lois, is a generalization of topological complexity and also Lusternik Schnirelmann category (cat). One of the importance of this new concept is to give easier proofs of the cat- and TC-related theorems due to the theorems in [6] which tells how the homotopic distance behaves under the composition. This leads us to think that new theorems on cat and TC can be proved with a help of homotopic distance.

In this section, we first give a brief background recalling the definitions of Lusternik Schnirelmann category, topological complexity, and homotopic distance and giving the relations between these concepts. Secondly, we talk about three properties of homotopic distance which allow us to understand the distance as an extended pseudometric. Later we introduce the topology induced by this pseudometric and give some topological properties of open balls $B_r(f)$'s.

In Section 2, we consider a specific space $\mathrm{Map}(S^1, S^1)$ where S^1 is a unit circle in a plane. One of the main results of this section is that open balls in this space are $B_r(f) = \{f\}$, if $r \leqslant 1$ and $\mathrm{Map}(S^1, S^1)$ otherwise. One another important result is that $\mathrm{Map}(S^1, S^1)$ is not compact. Later in the last section, we will combine these results with Čech closure operators and conclude that $\mathrm{Map}(X, Y)$ is not compact under some certain conditions.

In Section 3, we consider the general space $\mathrm{Map}(X, Y)$ and introduce some of its topological properties such as connectedness.

Let us recall that a map $f : X \to Y$ is called *null-homotopic*, provided f is homotopic to a constant map $c : X \to Y$ and also recall that an open covering of a topological space X is a collection of open subsets of X whose union is X.

Definition 1.1 [2] The Lusternik Schnirelmann category of a space X, $\mathrm{cat}(X)$, is the least non-negative integer $k \geqslant 0$, provided there exists an open covering $\{U_0, U_1, \ldots, U_k\}$ of X such that the inclusion on each U_i is null-homotopic for $i = 0, 1, \ldots, k$. If there is no such a covering, $\mathrm{cat}(X) = \infty$.

Recall that the fibration $\pi : PX \to X \times X$ which assigns a path γ in X its initial and final points is called a *path fibration*.

Definition 1.2 [4] Let $\pi : PX \to X \times X$, by $\pi(\gamma) = (\gamma(0), \gamma(1))$ be the path fibration. The topological complexity of a space X, $\mathrm{TC}(X)$, is the least non-negative integer $k \geqslant 0$, provided there exists an open covering $\{U_0, U_1, \ldots, U_k\}$ of $X \times X$ such that there exists a continuous section $s_i : U_i \to PX$ for each $i = 0, 1, \ldots, k$. If there is no such a covering, $\mathrm{TC}(X) = \infty$.

Definition 1.3 [6] Let $f, g : X \to Y$ be continuous maps. The homotopic distance between f and g, denoted by $\mathrm{D}(f, g)$, is the least non-negative integer k such that there exist open subsets $U_0, U_1, \cdots, U_k$ of X covering X satisfying $f\big|_{U_i} \simeq g\big|_{U_i}$ for each $i = 0, 1, \cdots, k$. If there is no such a covering, $\mathrm{D}(f, g) = \infty$.

The relations between homotopic distance D, cat and TC can be given as follows. We have $\mathrm{D}(id, c) = \mathrm{cat}(X)$ where id and c are the identity map and a constant map on X, respectively. We also have $\mathrm{D}(i_1, i_2) = \mathrm{cat}(X)$, provided $i_j : X \hookrightarrow X \times X$ for $j = 1, 2$, given by $i_1(x) = (x, x_0)$ and $i_2(x) = (x_0, x)$. Further, $\mathrm{D}(\mathrm{pr}_1, \mathrm{pr}_2) = \mathrm{TC}(X)$, provided $\mathrm{pr}_j : X \times X \to X$ is the projection to the j-th factor for $j = 1, 2$. For proofs and more details, we refer to [6].

The following three propositions listed below are the properties of homotopic distance which allow us to build a metric space. Also, The following proposition follows from the fact that homotopy is an equivalence relation.

Proposition 1.4 [6] If $f, g : X \to Y$ are maps, then $\mathrm{D}(f, g) = \mathrm{D}(g, f)$.

Proposition 1.5 [6] If $f, g : X \to Y$ are maps, then $\mathrm{D}(f, g) = 0$ iff $f \simeq g$.

The above proposition follows from the fact that if $f \simeq g$, we can take $U = X$ so $\mathrm{D}(f, g) = 0$, and vice versa.

Proposition 1.6 [6] If $f, g, h : X \to Y$ are maps and X is a normal space, then

$$\mathrm{D}(f, h) \leqslant \mathrm{D}(f, g) + \mathrm{D}(g, h)$$

Proof. This proof is given by Macías-Virgós and Mosquera-Lois in [6]. Let $\mathrm{D}(f, g) = m$ and $\mathrm{D}(g, h) = n$. Then there exists an open covering $\mathcal{U} = \{U_0, \ldots, U_m\}$ of X satisfying that $f|_{U_i} \simeq g|_{U_i}$ for each $i = 0, 1, \ldots, m$ and an open covering $\mathcal{V} = \{V_1, \ldots, V_n\}$ of X satisfying that $g|_{V_j} \simeq h|_{V_j}$ for each $j = 0, 1, \ldots, n$. Since these properties are closed for open subsets and disjoint unions, by [7, Lemma 4.3], X has an open covering $\mathcal{W} = \{W_0, \ldots, W_{m+n}\}$ satisfying that $f|_{W_k} \simeq g|_{W_k} \simeq h|_{W_k}$ for all $k = 0, 1, \ldots, m+n$. Therefore $\mathrm{D}(f, h) \leqslant m + n$. ■

A pseudometric d on a non-empty set M is a function $d : M \times M \to [0, \infty)$ that satisfies

M1) $d(m, m) = 0$,
M2) $d(m, n) = d(n, m)$,
M3) $d(m, n) \leqslant d(m, k) + d(k, n)$

for all $m, n, k \in M$. Further d is called a semimetric, provided it satisfies all but M3 with the additional condition that $d(m, n) = 0$ implies $m = n$. An extended pseudometric on M is a map $d : M \times M \to [0, \infty]$ satisfying the three axioms.

Let $\mathrm{Map}(X, Y)$ be the set of continuous maps from X to Y

$$\mathrm{Map}(X, Y) = \{f : X \to Y \mid f \text{ is continuous}\}.$$

Consider the following function

$$\begin{aligned} \mathrm{D} : \mathrm{Map}(X, Y) \times \mathrm{Map}(X, Y) &\to [0, \infty] \\ (f, g) &\mapsto \mathrm{D}(f, g). \end{aligned}$$

Notice that D is an extended semimetric on the quotient $\mathrm{Map}(X, Y)/\mathcal{R}$ where $\mathcal{R}$ is the equivalence relation on $\mathrm{Map}(X, Y)$ defined by

$$f \ \mathcal{R} \ g \ \text{ iff } \ \mathrm{D}(f, g) = 0.$$

We restrict ourselves for X to be a normal space so that D turns into an extended pseudometric on $\mathrm{Map}(X, Y)$. An (extended) pseudometric also induces an (extended) pseudometric space which is generated by the set of open balls. Then the topology on $\mathrm{Map}(X, Y)$ induced by D is generated by the open balls

$$B_r(f) := \{g \in \mathrm{Map}(X, Y) \mid \mathrm{D}(f, g) < r\}$$

for $r > 0$. Observe that $B_r(f)$ consists of maps which are homotopic to f for $r \leqslant 1$.

Remark 1 *For a simplicial complex K, the geometric realization $||K||$ is a normal*

Hausdorff space (see Theorem 17 in [9]). So, for simplicial complexes K and L, we can consider the topology on $\mathrm{Map}(||K||, ||L||)$ *which is induced by* D. *More generally, for a simplicial complex K and a topological space X, we can consider the topology on* $\mathrm{Map}(||K||, X)$ *induced by* D.

Throughout this paper, the 'domain space' X of $\mathrm{Map}(X, Y)$ is assumed to be a normal space.

Proposition 1.7 For $f \in \mathrm{Map}(X, Y)$, the open ball $B_r(f)$ has the indiscrete topology for $r \leqslant 1$.

Proof. We have $B_\varepsilon(f) \supseteq B_r(f)$ for all $f \in B_r(f)$ and $\varepsilon > 0$. ■

Proposition 1.7 yields that the open ball $B_r(f)$ is connected for $r \leqslant 1$. However this is not true for the case $r > 1$ under a certain condition.

Theorem 1.8 $B_r(f)$ is not connected for $r > 1$, provided $B_1(f)$ is a proper subset of $B_r(f)$.

Proof. Take $U = B_1(f)$. For $r > 1$, $U = B_1(f) \subseteq B_r(f)$. U is obviously open. Further, U is also closed, since its closure $\overline{U} = U \cup \{h \in B_r(f) \,|\, \mathrm{D}(h, U) = 0\}$ is equal to U where $\mathrm{D}(h, U) = \min_{g \in U} D(g, h)$.

Now take $V = U^c$ which is an open set. Since $B_1(f)$ is a proper subset of $B_r(f)$, V is non-empty. Hence U and V separates $B_r(f)$. This concludes that $B_r(f)$ is not connected. ■

Throughout this paper when we say 'space', we mean the pseudometric space induced by D.

2. $\mathrm{Map}(S^1, S^1)$

In this section, we consider a special case of $\mathrm{Map}(X, Y)$, that is, the space of continuous maps from S^1 to S^1. One of the main results in this section is that $\mathrm{Map}(S^1, S^1)$ is not compact and this result will be used in the proof of non-compactness of $\mathrm{Map}(X, Y)$ in the last section.

Let us start with the results which give the relation between the degree of a map and the homotopic distance.

Consider the space $\mathrm{Map}(S^1, S^1)$ where S^1 is the unit circle and let $f \in \mathrm{Map}(S^1, S^1)$. Then the induced map f_* from the fundamental group of the circle $\pi_1(S^1)$ to itself is a group homomorphism $f_* : \pi_1(S^1) \to \pi_1(S^1)$. Note that $\mathrm{Im}(f)$ is a subgroup of $\mathbb{Z}$ so that it is of the form $n\mathbb{Z}$ for some $n \in \mathbb{Z}$. This gives us that either f is a constant map or is of the form $z \mapsto z^n$ for $n \in \mathbb{Z}$.

Theorem 2.1 Consider the space $\mathrm{Map}(S^1, S^1)$. Let $f_n, f_m \in \mathrm{Map}(S^1, S^1)$ of degree n and m, respectively. Then $\mathrm{D}(f_n, f_m) = 1$.

Proof. We know that $\mathrm{D}(f_n, f_m) \leqslant \mathrm{cat}(S^1)$ by Corollary 3.9 in [6]. The fact that $\mathrm{cat}(S^1) = 1$ implies $\mathrm{D}(f_n, f_m) \leqslant 1$. Since the degrees of f_n and f_m are not equal, these maps cannot be homotopic. So $\mathrm{D}(f_n, f_m) \neq 0$. Hence $\mathrm{D}(f_n, f_m) = 1$. ■

Corollary 2.2 Consider the space $\mathrm{Map}(S^1, S^1)$ and the map f_n as described in Theorem 2.1 where $n \in \mathbb{Z}^+$. Then $D(f_n, c) = 1$, where c is any constant map $c : S^1 \to S^1$.

Proof. The constant map c and f_n are not homotopic and the proof follows similarly from Theorem 2.1. ■

For $f_n \in \mathrm{Map}(S^1, S^1)$ as described in Theorem 2.1, observe that $B_r(f_n) = \{f_n\}$ for $r \leqslant 1$ and $B_r(f_n) = \mathrm{Map}(S^1, S^1)$ for $r > 1$. Also $B_r(c) = \{\text{all constant maps on } S^1\}$ for $0 < r \leqslant 1$ and $B_r(c) = \mathrm{Map}(S^1, S^1)$ for $r > 1$.

Corollary 2.3 The space $\mathrm{Map}(S^1, S^1)$ is second countable (hence separable and Lindelöf).

Proof. A countable basis for $\mathrm{Map}(S^1, S^1)$ is $\mathcal{B} = \{\{f_n\} : n \in \mathbb{Z}\} \cup \{B_{\frac{1}{2}}(c)\}\}$. In a (extended) pseudometric space being second countable is equivalent with being separable and Lindelöf by [1, Lemma 17]. ■

Corollary 2.4 The space $\mathrm{Map}(S^1, S^1)$ is not compact.

Proof. Let f_n be the maps described in Theorem 2.1 and c be any constant map. Then the open cover $\mathcal{G} = \{\{f_n\} : n \in \mathbb{Z}\} \cup \{B_{\frac{1}{2}}(c)\}\}$ for $\mathrm{Map}(S^1, S^1)$ does not have a finite subcover. ■

3. Topological properties of Map(X, Y)

Lemma 3.1 Suppose X is an infinite discrete space. Then $\mathrm{D}(id_X, c) = \infty$ where id_X and c are the identity map and a constant map on X, respectively.

Proof. Any discrete space is normal so that D is a pseudometric on $\mathrm{Map}(X, X)$. Since X is discrete, id_X and c cannot be homotopic. Hence $\mathrm{D}(id_X, c) > 0$. Suppose $D(id_X, c) = n$ where n is a positive integer. By the definition of the homotopic distance, there exists an open cover $\mathcal{U} = \{U_0, U_1, \ldots, U_n\}$ for X such that $id_X\big|_{U_i} \simeq c\big|_{U_i}$ for $i = 0, 1, \ldots, n$. $id_X\big|_{U_i} \simeq c\big|_{U_i}$ implies that U_i is contractible in X, so that it is path connected. Since the only path connected subsets of a discrete space are singletons, $\mathcal{U}$ cannot be a cover for X. Thus $\mathrm{D}(id_X, c) = \infty$. ■

The space $\mathrm{Map}(X, Y)$ is not interesting whenever X or Y is contractible.

Theorem 3.2 If X or Y is contractible, then $\mathrm{Map}(X, Y)$ is indiscrete.

Proof. The proof follows from the fact that any two continuous maps from a contractible space to any space or vice versa are homotopic. ■

Proposition 3.3 If X is an infinite discrete space, then $\mathrm{Map}(X, X)$ is not path connected.

Proof. See Lemma 13 in [1]. ■

Theorem 3.4 $\mathrm{Map}(X, Y)$ is not connected, provided that $\mathrm{Map}(X, Y)$ is not indiscrete.

Proof. We can choose $U = B_1(f)$ for a fixed $f \in \mathrm{Map}(X, Y)$. Notice that U is non-empty, open and closed (see proof of Theorem 1.8). Define a set $V = \mathrm{Map}(X, Y) \setminus B_1(f)$. More precisely, there is a $g \in \mathrm{Map}(X, Y)$ which is not in $B_1(f)$. If we cannot find such a g, then we have $\mathrm{Map}(X, Y) = B_1(f)$ which contradicts with the fact that $\mathrm{Map}(X, Y)$ is not indiscrete. Hence V is non-empty and $\mathrm{Map}(X, Y)$ is not connected. ■

4. Non-compactness of Map(X, Y)

We begin this section with a brief introduction to Čech closure spaces on a set and on a pseudometric.

Definition 4.1 Let X be a set X and $P(X)$ denote its powerset. A Čech closure operator on X is a map $c : P(X) \to P(X)$ satisfying the following three axioms

(1) $c(\emptyset) = \emptyset$
(2) $A \subseteq c(A)$
(3) $c(A \cup B) = c(A) \cup c(B)$.

If c is a Čech closure operator on X, then the pair (X, c) is called a (Čech) closure space.

For a Čech closure space (X, c), the interior of a subset A of X is defined by $i_c(A) = X - c(X - A)$. A covering $\{U_i \,|\, i \in I\}$ of (X, c) is said to be an interior covering, provided $\cup_{i \in I} i_c(U_i) = X$.

The compactness for Čech closure spaces is given as follows.

Definition 4.2 [8] A Čech closure space (X, c) is said to be compact, provided every interior cover of a closure space (X, c) has a finite cover.

Definition 4.3 [8] For a metric space (X, d_X), $x \in X$, and $A \subset X$, the distance between x and A is given by $d(x, A) := \inf_{y \in A} d(x, y)$.

For $r \geqslant 0$, define a map $c_r : P(X) \to P(X)$ given by $c_r(A) = \{x \in X \mid d(x, A) \leqslant r\}$. Observe that c_r is a closure operator on the metric space (X, d_X) and c_0 is the topological closure operator on X for the topology induced by the metric [8].

Definition 4.4 [8] For a fixed $q, r > 0$, a map $f : (X, d_X) \to (Y, d_Y)$ is (q, r)-continuous if for every $\epsilon > 0$ and $x \in X$, there exists $\delta_x > 0$ such that

$$d_X(x, x') < q + \delta_x \quad \text{implies} \quad d_Y(f(x), f(x')) < r + \epsilon.$$

We know from [8, Proposition 3.5] that the (q, r)-continuity on metric spaces is equivalent to the continuity of maps between the associated closure spaces.

Proposition 4.5 [8] If (K, c_K) is a compact closure space, then a continuous map $f : (K, c_K) \to (X, c_r)$ has bounded image.

The proof of the above proposition as given by Reiser in [8] requires the following theorem which tells the continuity of a map between closure spaces.

Theorem 4.6 [3, Theorem 16.A.4 and Corollary 16.A.5] A map $f : (X, c_X) \to (Y, c_Y)$ between closure spaces is continuous at x if and only if for every neighbourhood $U \subseteq Y$ of $f(x)$, the inverse image $f^{-1}(U) \subseteq X$ is a neighbourhood of x.

Proof. [Proof of Proposition 4.5] This proof is given by Reiser in [8]. Fix an $\epsilon > 0$, consider the interior cover given by $\mathcal{U} = \{B_x\}_{x \in X}$ where

$$B_x := B(x, r + \epsilon) = \{y \in X \mid d(x, y) < r + \epsilon\}$$

is a neighbourhood of x. By Theorem 4.6, $\{f^{-1}(B_x)\}_{x \in X}$ is an interior cover of K. Compactness of K tells that $\{f^{-1}(B_x)\}_{x \in X}$ must have a finite subcover, say $\{f^{-1}(B_{x_i})\}_{i=1}^{n}$. Then the image of f is contained in $\cup_{i=1}^{n} B_{x_i}$, hence the image of f is bounded. ■

Theorem 4.7 If there is a continuous, surjective map $F : \mathrm{Map}(X,Y) \to \mathrm{Map}(S^1,S^1)$ such that $F^{-1}(\mathrm{Im}F)$ is finite, then $\mathrm{Map}(X,Y)$ is not compact.

Proof. Let us take the closure operators c_0 on both $\mathrm{Map}(X,Y)$ and $\mathrm{Map}(S^1,S^1)$. Assume that $\mathrm{Map}(X,Y)$ is compact. We will show that if F is continuous and $\mathrm{Im}F$ is not bounded, then we will obtain a contradiction. Hence we will conclude that $\mathrm{Map}(X,Y)$ is not compact. Let us assume that $\mathrm{Im}F$ is bounded. Then $\mathrm{Im}F \subseteq B$ where $B = \cup_{j=1}^{k} B_r(f_{m_j}) = \cup_{j=1}^{k}\{f_{m_j}\}$ for $r \leqslant 1$. Since $F^{-1}(\mathrm{Im}F)$ is finite and F is surjective, $\mathrm{Map}(X,Y)$ must be finite which is a contradiction since $\mathrm{Map}(S^1,S^1)$ is not compact by Corollary 2.4. ■

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