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On the continuity of some linear maps on certain Banach algebras

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Abstract. Let $\mathcal{A}$ and $\mathcal{U}$ be Banach algebras, θ be a nonzero character on $\mathcal{A}$ and let $\mathcal{A} \times_\theta \mathcal{U}$ be the corresponding Lau product Banach algebra. In this paper we investigate derivations and multipliers of $\mathcal{A} \times_\theta \mathcal{U}$ and study the automatic continuity of these maps. We also study continuity of the derivations for some special cases of $\mathcal{U}$ and the Banach $(\mathcal{A} \times_\theta \mathcal{U})$-bimodule $\mathcal{X}$ and establish various results in this respect. Some of the results are devoted to find conditions under which one can represent a derivation on $\mathcal{A} \times_\theta \mathcal{U}$ as sum of two derivations in such a way that one of them is continuous. Some examples are also given.

Keywords: Lau product, derivation, multiplier, automatic continuity.

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1. Introduction

Let $\mathcal{A}$ be a Banach algebra (over $\mathbb{C}$), and $\mathcal{X}$ be a Banach $\mathcal{A}$-bimodule. A linear map $D : \mathcal{A} \to \mathcal{X}$ is said to be a derivation if $D(ab) = aD(b) + D(a)b$ for all $a, b \in \mathcal{A}$. For any $x \in \mathcal{X}$, the map $\mathrm{ad}_x : \mathcal{A} \to \mathcal{X}$ given by $\mathrm{ad}_x(a) = ax - xa$ is a continuous derivation called inner. For a derivation $D : \mathcal{A} \to \mathcal{A}$, the corresponding generalized D-derivation is a linear map $\delta : \mathcal{X} \to \mathcal{X}$ satisfying $\delta(xa) = \delta(x)a + xD(a)$ for all $a \in \mathcal{A}$ and $x \in \mathcal{X}$.

One of the most well-known problems related to theory of derivations is to find some conditions which force them to be automatically continuous. This theory has been an active field of research during the recent decades, studied intensively by many authors from different points of view; (see for example, [7–13, 20, 21, 23, 25, 27]). For a full account on the subject, the reader can refer to [6] which is a comprehensive source in this context. A well-known result due to Johnson and Sinclair [16] asserts that every derivation on a

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semisimple Banach algebra is continuous. Ringrose [24] proved that every derivation from a C^*-algebra $\mathcal{A}$ into a Banach $\mathcal{A}$-bimodule $\mathcal{X}$ is continuous. The automatic continuity of derivations of some special classes of Banach algebras is studied by several authors (see [2, 5, 14, 28, 29]).

Another notion studied in the present paper is that of multiplier of Lau product $\mathcal{A} \times_\theta \mathcal{U}$. Recall that a multiplier on a Banach algebra $\mathcal{A}$ is a linear map $T : \mathcal{A} \to \mathcal{A}$ satisfying $aT(b) = T(a)b$ for all $a, b \in \mathcal{A}$. $\mathcal{A}$ is said to be faithful, if for any $x \in \mathcal{A}$, $\mathcal{A}x = \{0\} = x\mathcal{A}$ implies that $x = 0$. It is well-known and easy to show that if $\mathcal{A}$ is faithful, then every multiplier on $\mathcal{A}$ is continuous. The notion of multiplier was originally introduced by Helgason [15] and then was developed by Wang [30] and Birtal [3]. One can also refer to [17, 19].

Let $\mathcal{A}$ and $\mathcal{U}$ be two Banach algebras and $\theta : \mathcal{A} \to \mathbb{C}$ be a nontrivial character. If we consider the space $\mathcal{A} \times \mathcal{U}$ with the usual $\mathbb{C}$-module structure, the multiplication

$$(a, u)(a', u') = (aa', \theta(a)u' + \theta(a')u + uu'), \quad (a, a' \in \mathcal{A}, u, u' \in \mathcal{U})$$

with the norm

$$||(a, u)|| = ||a|| + ||u||,$$

turn $\mathcal{A} \times \mathcal{U}$ into a Banach algebra called Lau product Banach algebra, which is denoted by $\mathcal{A} \times_\theta \mathcal{U}$.

Lau Banach algebras were firstly introduced by T. Lau in [18] for special classes of Banach algebras that are predual of a von Neumann algebra, and subsequently developed by S. Monfared [22] in the general case.

Let $\mathcal{X}$ be a Banach $(\mathcal{A} \times_\theta \mathcal{U})$-bimodule. Then $\mathcal{X}$ is a Banach $\mathcal{A}$-bimodule by defining module operations in a natural fashion;

$$a \cdot x = (a, 0)x, \qquad x \cdot a = x(a, 0), \quad (a \in \mathcal{A}, x \in \mathcal{X}).$$

Similarly $\mathcal{X}$ turns into a Banach $\mathcal{U}$-bimodule via the module actions given by

$$u \cdot x = (0, u)x, \qquad x \cdot u = x(0, u) \quad (u \in \mathcal{U}, x \in \mathcal{X}).$$

For a linear map $T : \mathcal{X} \to \mathcal{Y}$ between two Banach spaces $\mathcal{X}$ and $\mathcal{Y}$, a key notion to study its continuity is the separating space $\mathfrak{S}(T)$ defined as

$$\mathfrak{S}(T) := \{y \in \mathcal{Y} \mid \text{ there is } \{x_n\} \subseteq \mathcal{X} \text{ with } x_n \to 0,\, T(x_n) \to y\}.$$

It is clear by the closed graph theorem that T is continuous if and only if $\mathfrak{S}(T) = \{0\}$.

Let $D : \mathcal{A} \to \mathcal{X}$ be a derivation. Then the two-sided continuity ideal of D is defined to be

$$\mathcal{I}(D) = \{a \in \mathcal{A} \,:\, a\mathfrak{S}(D) = \mathfrak{S}(D)a = 0\}.$$

Note that a derivation need not be continuous on $\mathcal{I}(D)$. But rather it is bounded as a bilinear form. However, if $\mathcal{I}(D)$ has a bounded approximate identity, then the restriction of D to its continuity ideal $\mathcal{I}(D)$ is continuous.

Let $\mathcal{A}$ be a Banach algebra and $\mathcal{X}$ and $\mathcal{Y}$ be Banach $\mathcal{A}$-bimodules. $Z(\mathcal{A})$, denotes the center of $\mathcal{A}$ and for $\mathcal{S} \subseteq \mathcal{X}$,

$$Z_{\mathcal{S}}(\mathcal{A}) = \{s \in \mathcal{S} : sa = as \, \text{for all} \, a \in \mathcal{A}\}.$$

Also, the annihilator of $\mathcal{A}$ over $\mathcal{S}$, denoted by $\text{ann}_{\mathcal{S}}\mathcal{A}$ is defined to be

$$\text{ann}_{\mathcal{S}}\mathcal{A} := \{s \in \mathcal{S} \, : \, s\mathcal{A} = \mathcal{A}s = \{0\}\}.$$

Similarly for a subset $\mathcal{D} \subseteq \mathcal{A}$ we write,

$$\text{ann}_{\mathcal{X}}\mathcal{D} := \{x \in \mathcal{X} \, : \, x\mathcal{D} = \mathcal{D}x = \{0\}\}.$$

This paper is organized as follows. In section 2, we shall focus on the derivations $D : \mathcal{A} \times_\theta \mathcal{U} \to \mathcal{X}$ and determine the general structure of them. Then, we obtain some conditions under which these maps are automatically continuous and establish various results in this context. Since inner derivations form an important class of automatically continuous derivations, some of the results are devoted to investigate the inner-ness of the derivations. Then in section 3, we apply our results to study the continuity of derivation $D : \mathcal{A} \times_\theta \mathcal{U} \to \mathcal{X}$ for some special cases of $\mathcal{U}$ and $\mathcal{X}$ and give some examples. We would also like to find conditions under which a derivation $D : \mathcal{A} \times_\theta \mathcal{U} \to \mathcal{A} \times_\theta \mathcal{U}$ can be represented as sum of a continuous derivation and another derivation. In section 4, we shall focus on the multipliers of Lau products and we obtain some results concerning continuity of them.

2. The derivations of $\mathcal{A} \times_\theta \mathcal{U}$

In this section we study the derivations of $\mathcal{A} \times_\theta \mathcal{U}$ and investigate continuity of them. Throughout, $\mathcal{A}, \mathcal{U}$ are Banach algebras, θ is a nonzero character on $\mathcal{A}$, $\mathcal{A} \times_\theta \mathcal{U}$ denotes the corresponding Lau product and $\mathcal{X}$ is a Banach $(\mathcal{A} \times_\theta \mathcal{U})$-bimodule. We commence this section with the following proposition characterizing the derivations from Lau Banach algebras into their Banach bimodules.

Proposition 2.1 Let $D : \mathcal{A} \times_\theta \mathcal{U} \to \mathcal{X}$ be a linear map. The following statements are equivalent.

(1) D is a derivation.

(2) There are linear maps $\delta_1 : \mathcal{A} \to \mathcal{X}$ and $\delta_2 : \mathcal{U} \to \mathcal{X}$ with

$$D(a, u) = \delta_1(a) + \delta_2(u) \quad (a \in \mathcal{A}, u \in \mathcal{U}),$$

such that δ_1 and δ_2 are derivations satisfying the following equation

$$a\delta_2(u) + \delta_1(a)u = \theta(a)\delta_2(u) = \delta_2(u)a + u\delta_1(a) \quad (a \in \mathcal{A}, u \in \mathcal{U}).$$

Proof. (1) $\Longrightarrow$ (2) Suppose that $D : \mathcal{A} \times_\theta \mathcal{U} \to \mathcal{X}$ is a derivation. Define linear maps δ_1 and δ_2 respectively by $\delta_1(a) = D(a, 0)$ and $\delta_2(u) = D(0, u)$. Hence $D(a, u) = \delta_1(a) + \delta_2(u)$. If we apply D on both sides of $(a, u)(a', u') = (aa', \theta(a)u' + \theta(a')u + uu')$, then it is easy to see that δ_1, δ_2 are derivations satisfying the above equation.

(2) $\Longrightarrow$ (1) Follows from a straightforward verification. ∎

In view of the above proposition, for any derivation $D : \mathcal{A} \times_\theta \mathcal{U} \to \mathcal{X}$, one can write $D = \delta_1 + \delta_2$ where δ_1, δ_2 are as in the proposition. We note that as inner derivations are continuous by definition, if we show that a given derivation D is inner, this implies that D is continuous. In the following result we describe inner derivations of $\mathcal{A} \times_\theta \mathcal{U}$.

Proposition 2.2 Let $D : \mathcal{A} \times_\theta \mathcal{U} \to \mathcal{X}$ be a derivation with $D = \delta_1 + \delta_2$. Then

(1) If D is inner, then δ_1 and δ_2 are inner.
(2) If $\delta_1 = \mathrm{ad}_{x_0}$ and $\delta_2 = \mathrm{ad}_{y_0}$, then $D = \mathrm{ad}_{z_0}$ for some $z_0 \in \mathcal{X}$ if and only if $z_0 - x_0 \in Z_{\mathcal{X}}(\mathcal{A})$ and $z_0 - y_0 \in Z_{\mathcal{X}}(\mathcal{U})$.

Proof.

(1) Since D is inner there exists some $x_0 \in \mathcal{X}$ for which $D = \mathrm{ad}_{x_0} = (a, u)x_0 - x_0(a, u)$ for all $a \in \mathcal{A}$ and $u \in \mathcal{U}$. If we substitute $a = 0$, we get $\delta_2(u) = ux_0 - x_0u$ for all $u \in \mathcal{U}$. Similarly if $u = 0$, then we have $\delta_1(a) = ax_0 - x_0a$ for all $a \in \mathcal{A}$. Thus δ_1 and δ_2 are inner.
(2) Suppose that $\delta_1 = \mathrm{ad}_{x_0}$ and $\delta_2 = \mathrm{ad}_{y_0}$ and there exists $z_0 \in \mathcal{X}$ such that $D = \mathrm{ad}_{z_0}$. We show that $z_0 - y_0 \in Z_{\mathcal{X}}(\mathcal{U})$. Note that

$$(a, u)z_0 - z_0(a, u) = ax_0 - x_0a + uy_0 - y_0u.$$

for all $a \in \mathcal{A}$ and $u \in \mathcal{U}$. Substituting $a = 0$ we have $u(z_0 - y_0) = (z_0 - y_0)u$ for all $u \in \mathcal{U}$. Thus $(z_0 - y_0) \in Z_{\mathcal{X}}(\mathcal{U})$. Analogously one can show that $z_0 - x_0 \in Z_{\mathcal{X}}(\mathcal{A})$. The converse is clear.

■

It is clear from the above proposition that if δ_1, δ_2 are inner derivations induced by the same element x_0 (i.e., $\delta_1 = \mathrm{ad}_{x_0}$ and $\delta_2 = \mathrm{ad}_{x_0}$), then D is always inner since one can take $z_0 = x_0$. However, it may happen that δ_1 and δ_2 are inner but D is not, as the following example shows.

***Example* 2.3** Consider that Banach algebra $\mathcal{A}$ of upper triangular 3×3 real matrices with 0 on the diagonal. So $\mathcal{A}^3 = 0$ but $\mathcal{A}^2 \neq 0$. Let a_0, b_0 be respectively a non-central and a central elements of $\mathcal{A}$. Define $\delta_1, \delta_2 : \mathcal{A} \to \mathcal{A}$ respectively by $\delta_1(a) = aa_0 - a_0a$ and $\delta_2(a) = ab_0 - b_0a = 0$. Then δ_1, δ_2 are inner but $D : \mathcal{A} \times_\theta \mathcal{A} \to \mathcal{A}$ with $D(a, b) = \delta_1(a) + \delta_2(b)$ is a derivation which is not inner. To show this, assume towards a contradiction that $D = \mathrm{ad}_{c_0}$ for some $c_0 \in \mathcal{A}$. Then by Proposition 2.2, $c_0 - a_0, c_0 - b_0 \in Z(\mathcal{A})$ and so $a_0 \in Z(\mathcal{A})$, a contradiction.

It is worthwhile to mention that the above example can be extended to a general case by considering $\mathcal{A}$ to be any non-commutative Banach algebra with $\mathcal{A}^3 = 0$.

The next result is a consequence of Proposition 2.1.

Proposition 2.4 Let $\mathcal{A}$ and $\mathcal{U}$ be Banach algebras and $\mathcal{X}$ be a Banach $(\mathcal{A} \times_\theta \mathcal{U})$-bimodule. Then

(1) Every derivation $\delta : \mathcal{A} \to \mathcal{X}$ with $\delta(\mathcal{A}) \subseteq \mathrm{ann}_{\mathcal{X}}\mathcal{U}$ extends to a derivation $\tilde{\delta} : \mathcal{A} \times_\theta \mathcal{U} \to \mathcal{X}$. In such a case, $\tilde{\delta}$ is continuous if and only if δ is continuous. Moreover, if $\delta = \mathrm{ad}_{x_0}$ for some $x_0 \in Z_{\mathcal{X}}(\mathcal{U})$, then so is $\tilde{\delta}$.
(2) Every derivation $D : \mathcal{U} \to \mathcal{X}$ gives rise to a derivation $\widetilde{D} : \mathcal{A} \times_\theta \mathcal{U} \to \mathcal{X}$. $\widetilde{D}$ is continuous if and only if D is continuous. Moreover, if $D = \mathrm{ad}_{x_0}$ for some $x_0 \in Z_{\mathcal{X}}(\mathcal{A})$, then so is $\widetilde{D}$.

Proof.

(1) It is clear that by the following module actions $\mathcal{X}$ turns into a Banach $(\mathcal{A} \times_\theta \mathcal{U})$-bimodule,

$$x \cdot (a, u) = xa, \qquad (a, u) \cdot x = ax,$$

for all $a \in \mathcal{A}, u \in \mathcal{U}$ and $x \in \mathcal{X}$. Define $\tilde{\delta} : \mathcal{A} \times_\theta \mathcal{U} \to \mathcal{X}$ by $\tilde{\delta}(a, u) = \delta(a)$. Since $\delta(\mathcal{A}) \subseteq \mathrm{ann}_{\mathcal{X}}\mathcal{U}$, we note that by Proposition 2.1 $\tilde{\delta}$ is a derivation. If δ is continuous, then so is $\tilde{\delta}$. Now if $\delta = \mathrm{ad}_{x_0}$ for some $x_0 \in Z_{\mathcal{X}}(\mathcal{U})$, then

$$\begin{aligned}\tilde{\delta}(a, u) &= (a, u)x_0 - x_0(a, u)\\ &= \mathrm{ad}_{x_0}(a, u) \qquad\qquad (a \in \mathcal{A}, u \in \mathcal{U}).\end{aligned}$$

(2) Suppose that $D : \mathcal{U} \to \mathcal{X}$ is a derivation. Then the module actions given by

$$x \cdot (a, u) = \theta(a)x + xu, \quad (a, u) \cdot x = \theta(a)x + ux, \quad ((a, u) \in \mathcal{A} \times \mathcal{U}, x \in \mathcal{X}),$$

makes $\mathcal{X}$ a Banach $(\mathcal{A} \times_\theta \mathcal{U})$-bimodule. Define $\widetilde{D} : \mathcal{A} \times_\theta \mathcal{U} \to \mathcal{X}$ by $\widetilde{D}(a, u) = D(u)$ for all $a \in \mathcal{A}, u \in \mathcal{U}$. So

$$\begin{aligned}\tilde{D}((a, u))(a', u')) &= \theta(a')D(u) + D(u)u' + \theta(a)D(u') + uD(u')\\ &= D(u)(a', u') + (a, u)D(u')\\ &= \tilde{D}((a, u))(a', u') + (a, u)\tilde{D}((a', u')), \qquad (a, a' \in \mathcal{A}, u, u' \in \mathcal{U}).\end{aligned}$$

implying that $\widetilde{D}$ is a derivation. Moreover, if $D = \mathrm{ad}_{x_0}$ for some $x_0 \in Z_{\mathcal{X}}(\mathcal{A})$,

$$\begin{aligned}\tilde{D}(a, u) &= D(u)\\ &= (a, u)x_0 - x_0(a, u) = \mathrm{ad}_{x_0}(a, u), \qquad (a \in \mathcal{A}, u \in \mathcal{U}).\end{aligned}$$

This completes the proof. ■

Note that in the preceding proposition the inner-ness of $\tilde{\delta}$ (resp. $\tilde{D}$) implies that of δ (resp. D). This follows directly from Proposition 2.2-(1). We now investigate the relationship between the separating spaces of δ_1, δ_2 and apply the results to study continuity of derivations.

Theorem 2.5 Let $D : \mathcal{A} \times_\theta \mathcal{U} \to \mathcal{X}$ be a derivation and δ_1, δ_2 be as in Proposition 2.1. Then

(1) $\mathfrak{S}(\delta_1)$ is an $\mathcal{A}$-subbimodule of $\mathcal{X}$ and $\mathfrak{S}(\delta_1) \subseteq \mathrm{ann}_{\mathcal{X}}\mathcal{U}$.
(2) $\mathfrak{S}(\delta_2)$ is a symmetric $\mathcal{A}$-subbimodule of $\mathcal{X}$ and $\mathfrak{S}(\delta_2) \subseteq Z_{\mathcal{X}}(\mathcal{A})$. Moreover $\mathfrak{S}(\delta_2)$ is a $\mathcal{U}$-subbimodule of $\mathcal{X}$, too.

Proof.

(1) We only prove the given inclusion. Let $x_0 \in \mathfrak{S}(\delta_1)$. Then there exists some sequence a_n in $\mathcal{A}$ such that $a_n \to 0$ and $\delta_1(a_n) \to x_0$. We have

$$\delta_2(u)a_n + u\delta_1(a_n) = \theta(a_n)\delta_2(u),$$

for all $u \in \mathcal{U}$. Letting n tend to infinity, we get $ux_0 = 0$ and similarly $x_0u = 0$ for all $u \in \mathcal{U}$. Hence $\mathfrak{S}(\delta_1) \subseteq \text{ann}_{\mathcal{X}}\mathcal{U}$.

(2) Let $y_0 \in \mathfrak{S}(\delta_2)$. By Proposition 2.1,

$$\delta_2(u_n)a + u_n\delta_1(a) = a\delta_2(u_n) + \delta_1(a)u_n,$$

for some sequence u_n in $\mathcal{U}$ with $u_n \to 0$ and all $a \in \mathcal{A}$. If we let n tend to infinity, we obtain $y_0a = ay_0$ for all $a \in \mathcal{A}$. Thus $\mathfrak{S}(\delta_2) \subseteq Z_{\mathcal{X}}(\mathcal{A})$.

■

Corollary 2.6 Suppose that $D : \mathcal{A} \times_\theta \mathcal{U} \to \mathcal{X}$ is a derivation with $D = \delta_1 + \delta_2$ where δ_1, δ_2 are as in Proposition 2.1. Then

(1) If either $\text{ann}_{\mathcal{X}}\mathcal{U} = \{0\}$ or $\mathcal{U}$ has a left (or right) bounded approximate identity for $\mathcal{X}$, then δ_1 is continuous.

(2) Suppose that $Z_{\mathcal{X}}(\mathcal{A}) = \{0\}$. Then $\delta_2 : \mathcal{U} \to \mathcal{X}$ is a continuous derivation. If in addition $\text{ann}_{\mathcal{X}}\mathcal{U} = \{0\}$, then every derivation $D : \mathcal{A} \times_\theta \mathcal{U} \to \mathcal{X}$ is continuous.

Proof.

(1) If $\text{ann}_{\mathcal{X}}\mathcal{U} = \{0\}$, by Theorem 2.5,

$$\mathfrak{S}(\delta_1) \subseteq \text{ann}_{\mathcal{X}}\mathcal{U} = \{0\}$$

so $\mathfrak{S}(\delta_1) = 0$ and therefore δ_1 is continuous.

(2) If $Z_{\mathcal{X}}(\mathcal{A}) = \{0\}$, then the inclusion given in part (2) of Theorem 2.5 implies $\mathfrak{S}(\delta_2) = \{0\}$. That is, δ_2 is continuous.

■

If $\mathcal{X}$ and $\mathcal{Y}$ are Banach $\mathcal{A}$ and $\mathcal{U}$-bimodules respectively, then it can be checked that the module actions

$$(a, u) \cdot x = ax \qquad , \qquad x \cdot (a, u) = xa$$

and

$$(a, u) \cdot y = \theta(a)y + uy \quad , \quad y \cdot (a, u) = \theta(a)y + yu, \quad (a \in \mathcal{A}, u \in \mathcal{U}, x \in \mathcal{X}, y \in \mathcal{Y})$$

turn $\mathcal{X}$ and $\mathcal{Y}$ into Banach $(\mathcal{A} \times_\theta \mathcal{U})$-bimodules. Now consider $\widetilde{\mathcal{M}} = \mathcal{X} \times \mathcal{Y}$. Observe that $\widetilde{\mathcal{M}}$ becomes a Banach $(\mathcal{A} \times_\theta \mathcal{U})$-bimodule with the module actions given by

$$(a, u) \cdot (x, y) = (ax, \theta(a)y + uy) \quad , \quad (x, y) \cdot (a, u) = (xa, \theta(a)y + yu),$$

for each $a \in \mathcal{A}, u \in \mathcal{U}, x \in \mathcal{X}, y \in \mathcal{Y}$.

Theorem 2.7 Let $\mathcal{X}, \mathcal{Y}$ be Banach $\mathcal{A}, \mathcal{U}$-bimodules respectively and $\widetilde{\mathcal{M}}$ defined as above. Then $D : \mathcal{A} \times_\theta \mathcal{U} \to \widetilde{\mathcal{M}}$ is a derivation if and only if

$$D(a, u) = D_{\mathcal{A}}(a) + D_{\mathcal{U}}(u), \quad (a \in \mathcal{A}, u \in \mathcal{U}),$$

where $D_{\mathcal{A}} : \mathcal{A} \to \mathcal{X}$ and $D_{\mathcal{U}} : \mathcal{U} \to \mathcal{Y}$ are derivations. Moreover, D is inner if and only if $D_{\mathcal{A}}, D_{\mathcal{U}}$ are inner in such a way that if $D = \text{ad}_z$ with $z = (x, y) \in \widetilde{\mathcal{M}}$, then $D_{\mathcal{A}} = \text{ad}_x$

and $D_{\mathcal{U}} = \mathrm{ad}_y$ and vice versa.

Proof. It can be routinely verified that D is a derivation if and only if $D_{\mathcal{A}}, D_{\mathcal{U}}$ are derivations. For the second part, suppose that $D = \mathrm{ad}_z$ for some $z = (x, y) \in \widetilde{\mathcal{M}}$. Then,

$$\begin{aligned} D(a,u) &= (a,u)(x,y) - (x,y)(a,u) \\ &= (ax - xa, uy - yu) \end{aligned}$$

for each $(a, u) \in \mathcal{A} \times_\theta \mathcal{U}$ and $(x, y) \in \widetilde{\mathcal{M}}$. It follows that $D_{\mathcal{A}}(a) = \mathrm{ad}_x(a)$ and $D_{\mathcal{U}}(u) = \mathrm{ad}_y(u)$. The reverse direction can be done in a similar way. ■

3. Continuity of derivations $D : \mathcal{A} \times_\theta \mathcal{U} \to \mathcal{X}$ in some special cases

Let $\mathcal{A}$, $\mathcal{U}$ and $\mathcal{X}$ be as in the previous section. In this subsection we shall study the continuity of the derivations $D : \mathcal{A} \times_\theta \mathcal{U} \to \mathcal{X}$ for some special cases of $\mathcal{X}$ and $\mathcal{U}$ and establish various results in this regard.

$\mathcal{X}$ is a simple Banach $\mathcal{A} \times_\theta \mathcal{U}$-bimodule

In this part we assume that $\mathcal{X}$ is a simple Banach $(\mathcal{A} \times_\theta \mathcal{U})$-bimodule and obtain some results as follows.

Theorem 3.1 Suppose that $\mathcal{X}$ is a simple Banach $(\mathcal{A} \times_\theta \mathcal{U})$-bimodule and $D : \mathcal{A} \times_\theta \mathcal{U} \to \mathcal{X}$ with $D = \delta_1 + \delta_2$ is any derivation. Then either δ_1 is continuous or $\mathrm{ann}_{\mathcal{X}}\mathcal{U} = \mathcal{X}$.

Proof. Since δ_1 is a derivation, by Theorem 2.5-(1), $\mathfrak{S}(\delta_1)$ is an $\mathcal{A}$-subbimodule of $\mathcal{X}$. Since $\mathcal{X}$ is simple, we have either $\mathfrak{S}(\delta_1) = \{0\}$ or $\mathfrak{S}(\delta_1) = \mathcal{X}$. The former clearly implies that δ_1 is continuous. If $\mathfrak{S}(\delta_1) = \mathcal{X}$, then by another application of the same theorem we conclude that $\mathrm{ann}_{\mathcal{X}}\mathcal{U} = \mathcal{X}$. ■

As an analogous result to the preceding theorem, we state the following.

Theorem 3.2 Let $\mathcal{X}$ be a simple Banach $(\mathcal{A} \times_\theta \mathcal{U})$-bimodule and $D : \mathcal{A} \times_\theta \mathcal{U} \to \mathcal{X}$ be a derivation with $D = \delta_1 + \delta_2$. Then either δ_2 is continuous or $\mathcal{X}$ is a symmetric Banach $\mathcal{A}$-bimodule.

Proof. $\mathfrak{S}(\delta_2)$ is an $\mathcal{U}$-subbimodule of $\mathcal{X}$. Since $\mathcal{X}$ is simple, so either $\mathfrak{S}(\delta_2) = \{0\}$ or $\mathfrak{S}(\delta_2) = \mathcal{X}$. The former implies that δ_2 is continuous. If $\mathfrak{S}(\delta_2) = \mathcal{X}$, then since $\mathfrak{S}(\delta_2) \subseteq Z_{\mathcal{X}}(\mathcal{A})$, then $Z_{\mathcal{X}}(\mathcal{A}) = \mathcal{X}$ or $\mathcal{A}\mathcal{X} = \mathcal{X}\mathcal{A}$. ■

We conclude from the above theorem that if $\mathcal{X}$ is a non-symmetric simple Banach $(\mathcal{A} \times_\theta \mathcal{U})$-bimodule, then δ_2 is automatically continuous.

As a consequence of Theorems 3.1 and 3.2, we state the following result.

Corollary 3.3 Suppose $\mathcal{X}$ is a simple Banach $(\mathcal{A} \times_\theta \mathcal{U})$-bimodule and $D : \mathcal{A} \times_\theta \mathcal{U} \to \mathcal{X}$ is a derivation with $D = \delta_1 + \delta_2$. Then D is continuous if either of the following conditions holds.

(1) $\mathrm{ann}_{\mathcal{X}}\mathcal{U} \neq \mathcal{X}$ and $Z_{\mathcal{X}}(\mathcal{A}) \neq \mathcal{X}$.
(2) $\mathrm{ann}_{\mathcal{X}}\mathcal{U} = \{0\}$ and $Z_{\mathcal{X}}(\mathcal{A}) \neq \mathcal{X}$.

Proof.

(1) It is clear by Theorems 3.1 and 3.2 that D is continuous.

(2) Follows from Theorem 3.2 and Corollary 2.6.

■

In this subsection we study derivations $D : \mathcal{A} \times_\theta \mathcal{A} \to \mathcal{X}$ where $\mathcal{A}$ is a Banach algebra and $\mathcal{X}$ a Banach $(\mathcal{A} \times_\theta \mathcal{A})$-bimodule.

In light of Corollary 2.6, if $Z_{\mathcal{X}}(\mathcal{A}) = \{0\}$, then since $\mathrm{ann}_{\mathcal{X}}\mathcal{A} \subseteq Z_{\mathcal{X}}(\mathcal{A})$, every derivation $D : \mathcal{A} \times_\theta \mathcal{A} \to \mathcal{X}$ is continuous.

To prove the next result, we state the following lemma.

Lemma 3.4 Let $D : \mathcal{A} \times_\theta \mathcal{A} \to \mathcal{X}$ be a derivation with $D = \delta_1 + \delta_2$. Then $\mathcal{I}(\delta_1) = \mathcal{A}$.

Proof. The result immediately follows from the definition of a continuity ideal with Theorem 2.5-(1). ■

Corollary 3.5 Let $D : \mathcal{A} \times_\theta \mathcal{A} \to \mathcal{X}$ be a derivation with $D = \delta_1 + \delta_2$. Then for any $a \in \mathcal{A}$, the linear map $d_a : \mathcal{A} \to \mathcal{X}$ given by $D_a(b) = a\delta_1(b)$ is a continuous derivation.

Proof. By Theorem 3.2 of [1] the continuity ideal $\mathcal{I}(D)$ of every derivation D coincides with the set $\{a \in \mathcal{A} \mid D_a \text{ is continuous}\}$ where $D_a(b) = a\delta_1(b)$. Now the result is clear by Lemma 3.4. ■

Let $\mathcal{A}$ and $\mathcal{U}$ be two Banach algebras. Then it is easy to see that by module actions

$$a \cdot u = \theta(a)u, \quad u \cdot a = \theta(a)u,$$

for all $a \in \mathcal{A}$ and $u \in \mathcal{U}$, $\mathcal{U}$ becomes a Banach $\mathcal{A}$-bimodule. Therefore in view of Proposition 2.1, δ_2 is a generalized δ_1-derivation satisfying

$$\delta_2(ua) = \delta_2(u)a + u\delta_1(a),$$

for all $a \in \mathcal{A}$ and $u \in \mathcal{U}$. The generalized derivation δ_2 appeared naturally in the decomposition of derivations $D : \mathcal{A} \times_\theta \mathcal{U} \to \mathcal{X}$ with $D = \delta_1 + \delta_2$. The next theorem connects the continuity of δ_1, δ_2 for the case $\mathcal{U} = \mathcal{A}$ to that of $\mathcal{A}$-bimodule homomorphisms.

Theorem 3.6 Suppose that $D : \mathcal{A} \times_\theta \mathcal{A} \to \mathcal{X}$ is a derivation with $D = \delta_1 + \delta_2$. Then δ_2 is a generalized δ_1-derivation if and only if $\delta_2 - \delta_1$ is a right $\mathcal{A}$-bimodule homomorphism.

Proof. First suppose that δ_2 is a generalized δ_1-derivation. Then we have

$$\begin{aligned}(\delta_2 - \delta_1)(ab) &= \delta_2(a)b + a\delta_1(b) - \delta_1(a)b - a\delta_1(b)\\ &= (\delta_2 - \delta_1)(a)\, b,\end{aligned}$$

for all $a, b \in \mathcal{A}$. Conversely, if $\delta_2 - \delta_1$ is a right $\mathcal{A}$-bimodule homomorphism, then by an easy calculation it can be seen that δ_2 is a generalized δ_1-derivation. ■

Remark 1 *Let us remark that a direct application of the Cohen factorization theorem shows that if $\mathcal{A}$ posseses a bounded approximate identity for $\mathcal{X}$, then every $\mathcal{A}$-bimodule homomorphism $\phi : \mathcal{A} \to \mathcal{X}$ is continuous. Indeed, let $(a_n) \subseteq \mathcal{A}$ be a sequence with $a_n \to 0$. Then by the Cohen factorization theorem there exist a sequence (b_n) in $\mathcal{A}$ converging to zero and some $c \in \mathcal{A}$ such that $a_n = cb_n$, so $\phi(a_n) = \phi(c)b_n \to 0$. Thus ϕ is continuous.*

Corollary 3.7 Let $\mathcal{A}$ be a Banach algebra with a bounded approximate identity and $D : \mathcal{A} \times_\theta \mathcal{A} \to \mathcal{X}$ be a derivation with $D = \delta_1 + \delta_2$. Then δ_1 is continuous if and only if δ_2 is continuous.

Proof. If $D : \mathcal{A} \times_\theta \mathcal{A} \to \mathcal{X}$ is a derivation given by $D = \delta_1 + \delta_2$, then it is clear that δ_2 satisfies $\delta_2(ab) = \delta_2(a)b + a\delta_2(b)$ and $\delta_2(ab) = \delta_2(a)b + a\delta_1(b)$ for all $a, b \in \mathcal{A}$. So, we have $\delta_2(a)b = \delta_1(a)b$ for all $a, b \in \mathcal{A}$. Hence $(\delta_2 - \delta_1)(\mathcal{A}) \subseteq \mathrm{ann}_{\mathcal{X}}\mathcal{A}$. ■

In the case where $\mathrm{ann}_{\mathcal{X}}\mathcal{A} = \{0\}$, δ_1, δ_2 agree on $\mathcal{A}$. For instance if $\mathcal{A}$ has a bounded approximate identity for $\mathcal{X}$, then $\mathrm{ann}_{\mathcal{X}}\mathcal{A} = \{0\}$.

Corollary 3.8 Let $D : \mathcal{A} \times_\theta \mathcal{A} \to \mathcal{X}$ be a derivation with $D = \delta_1 + \delta_2$. If $\mathrm{ann}_{\mathcal{X}}\mathcal{A} = \{0\}$, then $\delta_2 = \delta_1$. In this case any derivation $D : \mathcal{A} \times_\theta \mathcal{A} \to \mathcal{X}$ can be written as $D(a, b) = \delta_1(a) + \delta_1(b) = \delta_1(a + b)$.

The case $\mathcal{X} = \mathcal{U}$

As we noted before, $\mathcal{U}$ is an ideal in $\mathcal{A} \times_\theta \mathcal{U}$ and so $\mathcal{U}$ can be regarded as a Banach $(\mathcal{A} \times_\theta \mathcal{U})-$bimodule as well. The following proposition is a special case of Proposition 2.1.

Proposition 3.9 Let $\mathcal{A}$ and $\mathcal{U}$ be two Banach algebras. Then $D : \mathcal{A} \times_\theta \mathcal{U} \to \mathcal{U}$ is a derivation if and only if $D = \delta_1 + \delta_2$ such that $\delta_1 : \mathcal{A} \to \mathcal{U}$ and $\delta_2 : \mathcal{U} \to \mathcal{U}$ are derivations and

$$\theta(a)\delta_2(u) = \delta_1(a)u + a\delta_2(u) = u\delta_1(a) + \delta_2(u)a \quad (a \in \mathcal{A}, u \in \mathcal{U}).$$

In the next theorem we state some results similar to those of Theorem 2.5 and Corollary 2.6. Using the results we study the continuity of the derivations $D : \mathcal{A} \times_\theta \mathcal{U} \to \mathcal{U}$.

Theorem 3.10 Let $D : \mathcal{A} \times_\theta \mathcal{U} \to \mathcal{U}$ be a derivation with $D = \delta_1 + \delta_2$. Then

(1) $\mathfrak{S}(\delta_1)$ is an $\mathcal{A}$-subbimodule of $\mathcal{U}$ and $\mathfrak{S}(\delta_2)$ is an ideal in $\mathcal{U}$. In particular, $\mathcal{A}\mathfrak{S}(\delta_2) = \mathfrak{S}(\delta_2)A = \theta(\mathcal{A})\mathfrak{S}(\delta_2)$.
(2) $\mathfrak{S}(\delta_1)$ annihilates $\mathcal{U}$; that is, $\mathcal{U}\mathfrak{S}(\delta_1) = \mathfrak{S}(\delta_1)\mathcal{U} = \{0\}$.

The following corollary follows immediately from the above theorem.

Corollary 3.11 Suppose that $D : \mathcal{A} \times_\theta \mathcal{U} \to \mathcal{U}$ is a derivation with $D = \delta_1 + \delta_2$ and $\mathrm{ann}_{\mathcal{U}}\mathcal{U} = \{0\}$. Then δ_1 is continuous. In this case D is continuous if and only if δ_2 is continuous.

***Example* 3.12** Let H be a separable infinite-dimensional Hilbert space, $B(H)$ the algebra of bounded operators on H and $K(H)$ be the algebra of compact operators on H which is a closed ideal in $B(H)$. Then each derivation $D : B(H) \times_\theta K(H) \to K(H)$ is continuous since $K(H)$ is a C^*-algebra (hence possessing a bounded approximate identity) and Ringrose's theorem [24] guarantees the continuity of δ_2.

Note that any Banach algebra $\mathcal{U}$ with a bounded approximate identity satisfies the hypothesis of the above corollary; since in this case $\mathrm{ann}_{\mathcal{U}}\mathcal{U} = \{0\}$. In the case that $\mathcal{U}$ is semisimple, a well-known result of Johnson [16] implies the continuity of δ_2.

Proposition 3.13 Let $\mathcal{A}$ and $\mathcal{U}$ be Banach algebras such that $\mathcal{U}$ is semisimple. Then every derivation $D : \mathcal{A} \times_\theta \mathcal{U} \to \mathcal{U}$ is continuous.

Proof. Since $\mathcal{U}$ is semisimple, by the Johnson theorem [16], $\delta_2 : \mathcal{U} \to \mathcal{U}$ is continuous. On the other hand, $\mathrm{ann}_{\mathcal{U}}\mathcal{U} = \{0\}$, thus by Corollary 2.6-(1), δ_1 is continuous as well. Therefore every derivation $D : \mathcal{A} \times_\theta \mathcal{U} \to \mathcal{U}$ is continuous. ■

All C^*-algebras, semigroup algebras, measure algebras and unital simple algebras are semisimple Banach algebras with a bounded approximate identity. Thus the classes of Banach algebras satisfying the hypothesis of the above proposition is quite rich. Consequently, we have the following result.

Corollary 3.14 Suppose that $\mathcal{A}$ is a Banach algebra and $\mathcal{U}$ is a C^*-algebra. Then every derivation $D : \mathcal{A} \times_\theta \mathcal{U} \to \mathcal{U}$ is continuous.

The last case which will be discussed is the case where the Banach $(\mathcal{A} \times_\theta \mathcal{U})$-bimodule $\mathcal{X}$ is $\mathcal{A} \times_\theta \mathcal{U}$ itself.

The case $\mathcal{X} = \mathcal{A} \times_\theta \mathcal{U}$

Our aim in this part is to study derivations $D : \mathcal{A} \times_\theta \mathcal{U} \to \mathcal{A} \times_\theta \mathcal{U}$ and investigate the automatic continuity of them. We also find conditions under which D is expressible as sum of two derivations such that one of them is continuous. Note that as discussed in the comments after Corollary 3.5, $\mathcal{U}$ can be viewed as a Banach $\mathcal{A}$-bimodule. In the following theorem we determine the structure of the derivations on $\mathcal{A} \times_\theta \mathcal{U}$.

Theorem 3.15 Let $D : \mathcal{A} \times_\theta \mathcal{U} \to \mathcal{A} \times_\theta \mathcal{U}$ be a map. Then the following are equivalent:

(1) D is a derivation.

(2) $D(a, u) = (\delta_1(a) + \tau_1(u), \delta_2(a) + \tau_2(u))$, for all $a \in \mathcal{A}, u \in \mathcal{U}$ where

  (1) $\delta_1 : \mathcal{A} \to \mathcal{A}, \delta_2 : \mathcal{A} \to \mathcal{U}$ are derivations such that

$$\theta(\delta_1(a))u + \delta_2(a)u = 0 \quad \text{and} \quad \theta(\delta_1(a))u + u\delta_2(a) = 0, \quad (a \in \mathcal{A}, u \in \mathcal{U}).$$

  (2) $\tau_1 : \mathcal{U} \to \mathcal{A}$ is an $\mathcal{A}$-bimodule homomorphism such that $\tau_1(uu') = 0\,(u, u' \in \mathcal{U})$.

  (3) $\tau_2 : \mathcal{U} \to \mathcal{U}$ is a linear map satisfying

$$\tau_2(uu') = \theta(\tau_1(u))u' + \theta(\tau_1(u'))u + u\tau_2(u') + \tau_2(u)u', \qquad (u, u' \in \mathcal{U}).$$

  Also D is inner if and only if $\tau_1 = 0, \delta_2 = 0$, δ_1 and τ_2 are inner.

By the above theorem, for a derivation D on $\mathcal{A} \times_\theta \mathcal{U}$ we have

$$\delta_2(\mathcal{A}) \subseteq Z(\mathcal{U}), \quad \theta(a)\tau_1(u) = a\tau_1(u) = \tau_1(u)a,$$

and so $\tau_1(\mathcal{U}) \subseteq Z(\mathcal{A})$. Also $u\tau_1(u') + \tau_1(u)u' = 0$ for all $u, u' \in \mathcal{U}$ if and only if $\tau_1(\mathcal{U}) \subseteq \mathrm{Ker}\theta$. Additionally, $\delta_1(\mathcal{A}) \subseteq \mathrm{ann}_\mathcal{A}\mathcal{U} = \mathrm{Ker}\theta$ if and only if $\delta_2(\mathcal{A}) \subseteq \mathrm{ann}_\mathcal{U}\mathcal{U}$.

Corollary 3.16 Suppose that $\delta_1 : \mathcal{A} \to \mathcal{A}$, $\delta_2 : \mathcal{A} \to \mathcal{U}$, $\tau_1 : \mathcal{U} \to \mathcal{A}$ and $\tau_2 : \mathcal{U} \to \mathcal{U}$ are linear maps. Then

(1) $D : \mathcal{A} \times_\theta \mathcal{U} \to \mathcal{A} \times_\theta \mathcal{U}$ defined by $D(a, u) = (\delta_1(a), 0)$ is a derivation if and only if δ_1 is a derivation. Also, D is inner if and only if δ_1 is inner.

(2) $D : \mathcal{A} \times_\theta \mathcal{U} \to \mathcal{A} \times_\theta \mathcal{U}$ with $D(a, u) = (0, \delta_2(a))$ is a derivation if and only if δ_2 is a derivation and $\delta_2(A) \subseteq \mathrm{ann}_\mathcal{U}\mathcal{U}$. Moreover, if $\delta_2 = \mathrm{ad}_{u_0}$ is inner and $u_0 \in Z(\mathcal{U})$, then D is inner.

(3) $D : \mathcal{A} \times_\theta \mathcal{U} \to \mathcal{A} \times_\theta \mathcal{U}$ with $D(a, u) = (\tau_1(u), 0)$ is a derivation if and only if $\tau_1(uu') = 0, u\tau_1(u') + \tau_1(u)u' = 0\,(u.u' \in \mathcal{U})$. In this case D is inner if and only if $\tau_1 = 0$ and $Z(\mathcal{A}) \neq \emptyset$.

(4) $D : \mathcal{A} \times_\theta \mathcal{U} \to \mathcal{A} \times_\theta \mathcal{U}$ by $D(a, u) = (0, \tau_2(u))$ is (inner)derivation if and only if τ_2 is (inner) derivation.

If $\mathcal{A}$ and $\mathcal{U}$ are Banach algebras such that $\mathcal{A}$ is commutative, then by Thomas' theorem [26], for every derivation D on $\mathcal{A} \times_\theta \mathcal{U}$, $\delta_1(\mathcal{A}) \subseteq rad(\mathcal{A}) \subseteq \mathrm{Ker}\theta = \mathrm{ann}_{\mathcal{A}}\mathcal{U}$ and $\delta_2(\mathcal{A}) \subseteq \mathrm{ann}_{\mathcal{U}}\mathcal{U}$ (where δ_1, δ_2 are as in Theorem 3.15). Also, in this case $D = D_1 + D_2 + D_3$ where $D_1(a, u) = (\delta_1(a), 0)$, $D_2(a, u) = (0, \delta_2(a))$ and $D_3(a, u) = (\tau_1(u), \tau_2(u))$ are all derivations on $\mathcal{A} \times_\theta \mathcal{U}$.

It is clear from Theorem 3.15 that if $\tau_1 = 0$, then τ_2 becomes a derivation. Some conditions on $\mathcal{U}$ that force τ_1 to be the zero map are: $\mathcal{U}$ has a bounded approximate identity, $\mathcal{U}$ is unital and $\mathrm{ann}_{\mathcal{U}}\mathcal{U} = \{0\}$. For instance, if $\mathcal{U}$ is faithful, semisimple or any Banach algebra having an approximate identity, then $\mathrm{ann}_{\mathcal{U}}\mathcal{U} = \{0\}$. By Corollary 3.16-(4), the continuity of derivations on $\mathcal{A} \times_\theta \mathcal{U}$ implies the continuity of the derivations on $\mathcal{U}$. Particularly, if every derivation on $\mathcal{A} \times_\theta \mathcal{A}$ is continuous, then so is every derivation on $\mathcal{A}$.

Proposition 3.17 Let $\mathcal{A}$ and $\mathcal{U}$ be semisimple Banach algebras. Then every derivation $D : \mathcal{A} \times_\theta \mathcal{U} \to \mathcal{A} \times_\theta \mathcal{U}$ is continuous.

Proof. By [22], Theorem 3.1, $\mathcal{A} \times_\theta \mathcal{U}$ is semisimple if and only if both $\mathcal{A}$ and $\mathcal{U}$ are semisimple. Now the result follows from Johnson's theorem. ■

Corollary 3.18 Suppose that $\mathcal{A}$ is a C^*-algebra and $\mathcal{U}$ is a Banach algebra with a bounded approximate identity. Then every derivation on $\mathcal{A} \times_\theta \mathcal{U}$ is continuous if and only if every derivation on $\mathcal{U}$ is continuous.

Proof. First suppose that every derivation on $\mathcal{U}$ is continuous. Thus for every derivation $D : \mathcal{A} \times_\theta \mathcal{U} \to \mathcal{A} \times_\theta \mathcal{U}$ we have

$$D(a, u) = (\delta_1(a) + \tau_1(u), \delta_2(a) + \tau_2(u)),$$

such that δ_1, δ_2 are derivations and by Ringrose's result [24] are continuous. Since $\mathcal{U}$ has a bounded approximate identity, $\tau_1 = 0$ and τ_2 becomes a derivation. Thus D is continuous if and only if τ_2 is continuous. The converse is clear. ■

Remark 2 *Let $\mathcal{A}$ be a commutative and $\mathcal{U}$ a semisimple Banach algebra with $\mathcal{U}$ having a bounded approximate identity. Then every derivation $D : \mathcal{A} \times_\theta \mathcal{U} \to \mathcal{A} \times_\theta \mathcal{U}$ is of the form $D = D_1 + D_2$ where $D_1(a, u) = (\delta_1(a), 0)$ and $D_2(a, u) = (0, \tau_2(u))$ are derivations on $\mathcal{A} \times_\theta \mathcal{U}$ such that D_2 is continuous. In particular, in this situation if every derivation on $\mathcal{A}$ is continuous, then every derivation on $\mathcal{A} \times_\theta \mathcal{U}$ is continuous.*

Theorem 3.19 Let $\mathcal{A}$ and $\mathcal{U}$ be Banach algebras with bounded approximate identities. If $\mathcal{A}$ is commutative, then every derivation $D : \mathcal{A} \times_\theta \mathcal{U} \to \mathcal{A} \times_\theta \mathcal{U}$ can be written as $D = D_1 + D_2$ where $D_1(a, u) = (\delta_1(a), \tau_2(u))$ and $D_2(a, u) = (0, \delta_2(a))$ are derivations on $\mathcal{A} \times_\theta \mathcal{U}$. Moreover, D_1 is continuous if one of the following conditions holds.

(1) There exists a surjective $\mathcal{A}$-module homomorphism $\phi : \mathcal{A} \to \mathcal{U}$ and δ_1 is continuous.

(2) There exists an injective $\mathcal{A}$-module homomorphism $\phi : \mathcal{A} \to \mathcal{U}$ and τ_2 is continuous.

Proof.

(1) Define $\psi : \mathcal{A} \to \mathcal{U}$ by $\psi = \tau_2 \circ \phi - \phi \circ \delta_1$. It is easy to see that ψ is a continuous left $\mathcal{A}$-module homomorphism. Similarly, $\phi \circ \delta_1$ is continuous and so is $\tau_2 \circ \phi$. On

the other hand since ϕ is surjective, $\mathfrak{S}(\tau_2 \circ \phi) = \mathfrak{S}(\tau_2) = \{0\}$ by [6, Proposition 5.2.2]. Thus τ_2 is continuous.

(2) The proof is similar to part (1).

■

Note that if $\delta : \mathcal{A} \to \mathcal{U}$ is a continuous derivation, the condition $\delta(\mathcal{A}) \subseteq \mathrm{ann}_{\mathcal{U}}\mathcal{U}$ is not satisfied in general, as the following example shows.

***Example* 3.20** Assume that G is a non-discrete abelian group. It has been shown in [4] that there is a nonzero continuous point derivation d at a nonzero character θ on $M(G)$. $\mathbb{C}$ turns into a symmetric banach $M(G)$-bimodule, if it is endowed with the following module actions:

$$c \cdot \mu = \theta(\mu)c \quad , \quad \mu \cdot c = \theta(\mu)c \quad (c \in \mathbb{C}, \mu \in M(G))$$

Now consider $M(G) \times_\theta \mathbb{C}$. Every derivation from $M(G)$ into $\mathbb{C}_\theta$ is a point derivation at θ. It is clear that $\mathrm{ann}_{\mathbb{C}}\mathbb{C} = \{0\}$. But $d \in Z^1(M(G), \mathbb{C}_\theta)$ is a nonzero derivation such that $d(M(G)) \nsubseteq \mathrm{ann}_{\mathbb{C}}\mathbb{C} = \{0\}$.

4. The multipliers on $\mathcal{A} \times_\theta \mathcal{U}$

In this section we turn our attention to the multipliers of Lau products. As before, $\mathcal{A}$,$\mathcal{U}$ are Banach algebras, $\theta \in \Delta(\mathcal{A})$ is a nonzero character and $\mathcal{A} \times_\theta \mathcal{U}$ denotes the associated Lau Banach algebra.

In the following theorem we characterize the multipliers on $\mathcal{A} \times_\theta \mathcal{U}$.

Theorem 4.1 A linear map $T : \mathcal{A} \times_\theta \mathcal{U} \to \mathcal{A} \times_\theta \mathcal{U}$ is a multiplier if and only if there are some linear maps $R_1 : \mathcal{A} \to \mathcal{A}$, $R_2 : \mathcal{A} \to \mathcal{U}$, $S_1 : \mathcal{U} \to \mathcal{A}$ and $S_2 : \mathcal{U} \to \mathcal{U}$ with

$$T(a, u) = (R_1(a) + S_1(u), R_2(a) + S_2(u)), \qquad (a \in \mathcal{A}, u \in \mathcal{U}),$$

satisfying the following conditions:

(1) $R_1 : \mathcal{A} \to \mathcal{A}$ is a multiplier.
(2) $aS_1(u) = S_1(u)a = 0$.
(3) $\theta(a)R_2(a') = \theta(a')R_2(a)$.
(4) $\theta(a)S_2(u) = \theta(R_1(a))u + R_2(a)u = \theta(R_1(a))u + uR_2(a)$.
(5) $\theta(S_1(u))u' + S_2(u)u' = \theta(S_1(u'))u + uS_2(u')$.

for all $a, a' \in \mathcal{A}$ and $u, u' \in \mathcal{U}$.

Proof. First suppose that T is a multiplier. Since T is linear, there exist some linear maps $R_1 : \mathcal{A} \to \mathcal{A}$, $R_2 : \mathcal{A} \to \mathcal{U}$, $S_1 : \mathcal{U} \to \mathcal{A}$ and $S_2 : \mathcal{U} \to \mathcal{U}$ with

$$T((a, u)) = (R_1(a) + S_1(u), R_2(a) + S_2(u))$$

for all $a \in \mathcal{A}$ and $u \in \mathcal{U}$. By the definition, $T((a, u))(a', u') = (a, u)T((a', u'))$ For all $a, a' \in \mathcal{A}$ and $u, u' \in \mathcal{U}$. If we substitute $u = u' = 0$, then we deduce that R_1 is a multiplier and $\theta(a)R_2(a') = \theta(a')R_2(a)$ for all $a, a' \in \mathcal{A}$. Similarly, substituting $a = a' = 0$ yields (v). If we put $a' = 0, u = 0$ and $a = 0, u' = 0$ respectively, we obtain equalities given in (4). Also putting $a' = u' = 0$, we conclude that $aS_1(u) = S_1(u)a = 0$ for all $a \in \mathcal{A}$ and $u \in \mathcal{U}$. The converse is straightforward and is left for the reader. ■

In view of the above theorem, in the sequel we can consider any multiplier $T : \mathcal{A} \times_\theta \mathcal{U} \to \mathcal{A} \times_\theta \mathcal{U}$ as

$$T((a,u)) = (R_1(a) + S_1(u), R_2(a) + S_2(u)), \qquad (a \in \mathcal{A}, u \in \mathcal{U}),$$

in which the mentioned maps satisfy conditions $(1) - (5)$. Part (4) of the above theorem implies that R_2 is a center-valued map; that is, it always maps $\mathcal{A}$ into the center of $\mathcal{U}$ (i.e., $R_2(\mathcal{A}) \subseteq Z(\mathcal{U})$) and also if we put $u = u'$ in (v), we conclude that $S_2(u)u = uS_2(u)$ for all $u \in \mathcal{U}$. Moreover by (2), $S_1(\mathcal{A}) \subseteq \mathrm{ann}_{\mathcal{A}}\mathcal{A}$.

We now state the following theorem.

Theorem 4.2 Suppose that $T : \mathcal{A} \times_\theta \mathcal{U} \to \mathcal{A} \times_\theta \mathcal{U}$ is a multiplier with

$$T(a,u) = (R_1(a) + S_1(u), R_2(a) + S_2(u)), \qquad (a \in \mathcal{A}, u \in \mathcal{U}).$$

Then

(1) The maps $R_2 : \mathcal{A} \to \mathcal{U}$ and $S_2 : \mathcal{U} \to \mathcal{U}$ are automatically continuous.
(2) $\mathfrak{S}(R_1), \mathfrak{S}(S_1) \subseteq \mathrm{Ker}(\theta)$.

Proof.

(1) Let $s \in \mathfrak{S}(S_2)$. Thus there exists some u_n in $\mathcal{U}$ for which $u_n \to 0$ and $S_2(u_n) \to s$. By Theorem 4.1-(4), we have

$$\theta(a)S_2(u_n) = \theta(R_1(a))u_n + R_2(a)u_n$$

for all $a \in \mathcal{A}$. Letting n tend to infinity, we obtain $\theta(a)s = 0$ for all $a \in \mathcal{A}$. So $s = 0$ and hence S_2 is continuous. For the continuity of R_2, suppose that $s' \in \mathfrak{S}(R_2)$ and a_n is a sequence in $\mathcal{A}$ with $a_n \to 0$ and $R_2(a_n) \to s'$. By part (2) of the preceding theorem we have that $\theta(a')R_2(a_n) = \theta(a_n)R_2(a')$ for all $a' \in \mathcal{A}$. By taking limits one obtains $s' = 0$. Thus R_2 is continuous.

(2) Let $a_0 \in \mathfrak{S}(R_1)$. Then there exists $a_n \subseteq \mathcal{A}$ such that $a_n \to 0$ and $R_1(a_n) \to a_0$. Therefore,

$$\theta(a_n)S(u) = \theta(R_1(a_n))u + R_2(a_n)u$$

for all $y \in \mathcal{U}$. So $\theta(a_0)u = 0$ for all $u \in \mathcal{U}$. Thus, $a_0 \in \mathrm{Ker}\theta$. The other inclusion is similar.

■

If $\mathcal{A}$ is a faithful Banach algebra, then every multiplier on $\mathcal{A}$ is continuous. On the other hand, in this case $S_1 = 0$, since by Theorem 4.1-(2), $S_1(\mathcal{A}) \subseteq \mathrm{ann}_{\mathcal{A}}\mathcal{A} = \{0\}$, so $S_2 : \mathcal{U} \to \mathcal{U}$ is then a multiplier on $\mathcal{U}$. Therefore we deduce the following result.

Theorem 4.3 Suppose that $\mathcal{A}$ and $\mathcal{U}$ are Banach algebras where $\mathcal{A}$ is faithful. Then every multiplier on $\mathcal{A} \times_\theta \mathcal{U}$ is continuous.

It is well-known that in each of the following cases the Banach algebra $\mathcal{A}$ is faithful:

(1) $\mathcal{A}$ is unital.
(2) $\mathcal{A}$ has an approximate identity (for example, $\mathcal{A}$ is a C^*-algebra).
(3) $\mathcal{A}$ is semiprime.
(4) $\mathcal{A}$ is a semisimple.

If $\mathcal{A}$ and $\mathcal{U}$ are as in the preceding theorem, an easy calculation shows that

$$\mathrm{ann}_{(\mathcal{A}\times_\theta\mathcal{U})}(\mathcal{A}\times_\theta\mathcal{U}) = \{(0,u) : u \in \mathrm{ann}_{\mathcal{U}}\mathcal{U}\} \cong \mathrm{ann}_{\mathcal{U}}\mathcal{U}.$$

Therefore, if $\mathcal{A}$ is faithful, $\mathcal{A}\times_\theta\mathcal{U}$ is faithful if and only if $\mathcal{U}$ is also. If we assume that $\mathcal{U}$ is not faithful, then $\mathcal{A}\times_\theta\mathcal{U}$ is not faithful as well whereas by Theorem 4.3, all multipliers $T : \mathcal{A}\times_\theta\mathcal{U} \to \mathcal{A}\times_\theta\mathcal{U}$ are continuous. This result can be interesting on its own as it can provide non-faithful Banach algebras on which every multiplier is continuous.

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