Abstract. Here, a new certain class of contractive mappings in the $b$-metric spaces is introduced. Some fixed point theorems are proved which generalize and modify the recent results in the literature. As an application, some results in the $b$-metric spaces endowed with a partial ordered are proved.

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1. Introduction

The existence of a fixed point is studied by many authors. The notion of $b$-metric space was first explained by Bakhtin in [2] and then widely utilized by Czerwik in [6] (this space is a metric type spaces defined by Khamsi and Hussain [18]). Since then, many researches deal with fixed point theory for single-valued and multi-valued mappings in $b$-metric spaces (see, [3, 6, 7] and references therein). Meanwhile, Samet et al. [30] presented the notions of $\alpha$-$\psi$-contractive and $\alpha$-admissible mappings and founded several fixed point theorems for such mappings outline under the complete metric spaces. Subsequently, Salimi et al. [28] and Hussain et al. [13] improved the concepts of $\alpha$-$\psi$-contractive and $\alpha$-admissible mappings and studied some fixed point theorems. In this paper, a new classes of contractive mappings is introduced in order to study some fixed point theorems in the $b$-metric spaces.

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Definition 1.1 [6] Let $X$ be a nonempty set and $s \geq 1$. A function $d : X \times X \to \mathbb{R}^+$ is a $b$–metric if and only if for all $x, y, z \in X$, the following conditions hold:

(b$_1$) $d(x, y) = 0$ iff $x = y$,
(b$_2$) $d(x, y) = d(y, x)$,
(b$_3$) $d(x, z) \leq s[d(x, y) + d(y, z)]$.

Then the tripled $(X, d, s)$ is called a $b$–metric space.

Definition 1.2 [5] Let $(X, d)$ be a $b$–metric space. A sequence $\{x_n\}$ in $X$ is called:

(a) $b$–convergent if and only if there exists $x \in X$ such that $d(x_n, x) \to 0$, as $n \to +\infty$. In this case, we write $\lim_{n \to \infty} x_n = x$.

(b) $b$–Cauchy if and only if $d(x_n, x_m) \to 0$, as $n, m \to +\infty$.

Proposition 1.3 [5, Remark 2.1] In a $b$–metric space $(X, d)$ the following assertions hold:

$p_1$. A $b$–convergent sequence has a unique limit.

$p_2$. Each $b$–convergent sequence is $b$–Cauchy.

$p_3$. In general a $b$–metric is not continuous.

Lemma 1.4 [1] Let $(X, d)$ be a $b$–metric space with $s \geq 1$. Suppose that $\{x_n\}$ and $\{y_n\}$ are $b$–convergent to $x, y$, respectively. Then

$$\frac{1}{s^2}d(x, y) \leq \liminf_{n \to \infty} d(x_n, y_n) \leq \limsup_{n \to \infty} d(x_n, y_n) \leq s^2d(x, y).$$

In particular, if $x = y$ then $\lim_{n \to \infty} d(x_n, y_n) = 0$. Moreover, for each $z \in X$

$$\frac{1}{s}d(x, z) \leq \liminf_{n \to \infty} d(x_n, z) \leq \limsup_{n \to \infty} d(x_n, z) \leq sd(x, z).$$

For more details on $b$-metric spaces the reader can refer to [7]-[11].

Definition 1.5 [30] Let $T$ be a self-mapping on $X$ and $\alpha : X \times X \to [0, +\infty)$ be a function. $T$ is an $\alpha$-admissible mapping if

$x, y \in X, \quad \alpha(x, y) \geq 1 \quad \Rightarrow \quad \alpha(Tx, Ty) \geq 1$.

Definition 1.6 [16] Let $T$ be an $\alpha$-admissible mapping. We say that $T$ is a triangular $\alpha$-admissible mapping if $\alpha(x, y) \geq 1$ and $\alpha(y, z) \geq 1$ implies that $\alpha(x, z) \geq 1$.

Lemma 1.7 [16] Let $T$ be a triangular $\alpha$-admissible mapping. Assume that there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$. Define sequence $\{x_n\}$ by $x_n = T^n x_0$. Then

$\alpha(x_m, x_n) \geq 1$ for all $m, n \in \mathbb{N}$ with $m < n$.

Definition 1.8 [12] Let $\alpha : X \times X \to [0, +\infty)$ and $T : X \to X$. We say that $T$ is an $\alpha$-continuous mapping if for given $x \in X$ and sequence $\{x_n\}$ with $x_n \to x$ and $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ one has $Tx_n \to Tx$.

Definition 1.9 Let $T$ be a self-mapping on $X$ and let $\lambda : X \to [0, +\infty)$ be a function. We say that $T$ is a semi $\lambda$-subadmissible mapping if

$x \in X, \quad \lambda(x) \leq 1 \quad \Rightarrow \quad \lambda(Tx) \leq 1$. 
Example 1.10 Let \( T : \mathbb{R} \to \mathbb{R} \) be defined by \( T x = x^3 \). Suppose that \( \lambda : \mathbb{R} \to \mathbb{R}^+ \) is given by \( \lambda(x) = e^x \) for all \( x \in \mathbb{R} \). Then \( T \) is a semi-\( \lambda \)-subadmissible mapping. Indeed, if \( \lambda(x) = e^x \leq 1 \) then \( x \leq 0 \) which implies that \( T x \leq 0 \). Therefore \( \lambda(T x) = e^{T x} \leq 1 \).

Consistent with Khan et al. [17] we denote by \( \Psi \) the set of all function \( \varphi : [0, +\infty) \to [0, +\infty) \) (which is called an altering distance function) if the following conditions hold:

- \( \varphi \) is continuous and non-decreasing.
- \( \varphi(t) = 0 \) if and only if \( t = 0 \).

Motivated by Kumam and Roldán [20] we introduce the following class of mappings which is suitable for our results.

Let \( \Theta \) denote the set of all functions \( \theta : R^4_+ \to R^+ \) satisfying:

- \((\Theta_1)\) \( \theta \) is continuous and increasing in all its variables;
- \((\Theta_2)\) \( \theta(t_1, t_2, t_3, t_4) = 0 \) iff either \( t_1 = 0 \) or \( t_4 = 0 \).

2. Main Theorems

In this section we state the Main results. The first theorem is based on [7, Theorem 4] and [27, Theorem 3].

Theorem 2.1 Let \((X, d, s)\) be a complete \( b \)-metric space, \( T \) be a self-mapping on \( X \) and \( \alpha : X \times X \to [0, \infty) \) and \( \lambda : X \to [0, +\infty) \) be two functions. Suppose that the following assertions hold.

(i) There exists \( x_0 \in X \) such that \( \alpha(x_0, Tx_0) \geq 1 \) and \( \lambda(x_0) \leq 1 \).
(ii) \( T \) is \( \alpha \)-continuous, triangular \( \alpha \)-admissible and semi \( \lambda \)-subadmissible mapping.
(iii) For all \( x, y \in X \) with \( \alpha(x, y) \geq 1 \)

\[
\psi(s d(Tx, Ty)) \leq \lambda(x) \lambda(y) \left[ \psi(M(x, y)) - \varphi(M(x, y)) \right] + \theta \left( d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx) \right)
\]

where \( \psi, \varphi \in \Psi \), \( \theta \in \Theta \) and

\[
M(x, y) = \max \left\{ d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty) + d(y, Tx)}{2s} \right\}.
\]

Then \( T \) has a fixed point.

Proof. Let \( x_0 \in X \) be such that \( \alpha(x_0, Tx_0) \geq 1 \) and \( \lambda(x_0) \leq 1 \). We define a sequence \( \{x_n\} \) as follows

\[
x_n = T^n x_0 = Tx_{n-1}
\]

for all \( n \in \mathbb{N} \). If \( x_n = x_{n+1} \) for some \( n \in \mathbb{N} \) then \( x_n = Tx_n \) and so \( x_n \) is a fixed point of \( f \). Hence we assume that \( x_n \neq x_{n+1} \), for all \( n \in \mathbb{N} \). Since \( T \) is a triangular \( \alpha \)-admissible mapping then by Lemma 1.7

\[
\alpha(x_m, x_n) \geq 1 \text{ for all } m, n \in \mathbb{N} \text{ with } m < n.
\]

Also, since \( T \) is a semi \( \lambda \)-subadmissible mapping and \( \lambda(x_0) \leq 1 \) then \( \lambda(x_1) = \lambda(Tx_0) \leq 1 \). Again, since \( T \) is semi \( \lambda \)-subadmissible, then \( \lambda(x_2) = \lambda(Tx_1) \leq 1 \). Continuing this process.
\[ \lambda(x_n) \leq 1 \] for all \( n \in \mathbb{N} \cup \{0\} \). Then by (iii),

\[
\psi(d(x_n, x_{n+1})) \leq \psi(sd(x_n, x_{n+1})) = \psi(sd(Tx_{n-1}, Tx_n)) \\
\leq \lambda(x_{n-1}) \lambda(x_n) \left[ \psi(M(x_{n-1}, x_n)) - \varphi(M(x_{n-1}, x_n)) \right] \\
+ \theta(d(x_{n-1}, Tx_{n-1}), d(x_n, Tx_n), d(x_{n-1}, Tx_n), d(x_n, Tx_{n-1})) \\
\leq \psi(M(x_{n-1}, x_n)) - \varphi(M(x_{n-1}, x_n)) \\
+ \theta(d(x_{n-1}, Tx_{n-1}), d(x_n, Tx_n), d(x_{n-1}, Tx_n), d(x_n, Tx_{n-1}))
\] (2)

where

\[
M(x_{n-1}, x_n) = \max \left\{ d(x_{n-1}, x_n), d(x_{n-1}, Tx_{n-1}), d(x_n, Tx_n), \frac{d(x_{n-1}, Tx_n) + d(x_n, Tx_{n-1})}{2s} \right\}
\]

\[
= \max \left\{ d(x_{n-1}, x_n), d(x_n, x_{n+1}), \frac{d(x_{n-1}, x_{n+1})}{2s} \right\}
\]

\[
\leq \max \left\{ d(x_{n-1}, x_n), d(x_n, x_{n+1}), \frac{sd(x_{n-1}, x_n) + sd(x_n, x_{n+1})}{2s} \right\}
\]

\[
= \max \left\{ d(x_{n-1}, x_n), d(x_n, x_{n+1}), \frac{d(x_{n-1}, x_{n+1}) + d(x_n, x_{n+1})}{2} \right\}
\]

(3)

and

\[
\theta(d(x_{n-1}, Tx_{n-1}), d(x_n, Tx_n), d(x_{n-1}, Tx_n), d(x_n, Tx_{n-1})) = \theta(d(x_{n-1}, x_n), d(x_n, x_{n+1}), d(x_{n-1}, x_{n+1}), d(x_n, x_n))
\]

\[
= \theta(d(x_{n-1}, x_n), d(x_n, x_{n+1}), d(x_{n-1}, x_{n+1}), 0) = 0.
\] (4)

By (2)-(4) and the properties of \( \psi \) and \( \varphi \) we obtain

\[
\psi(d(x_n, x_{n+1})) \leq \psi \left( \max \left\{ d(x_{n-1}, x_n), d(x_n, x_{n+1}) \right\} \right) - \varphi \left( M(x_{n-1}, x_n) \right)
\]

\[
< \psi \left( \max \left\{ d(x_{n-1}, x_n), d(x_n, x_{n+1}) \right\} \right). 
\] (5)

Now if

\[
\max \left\{ d(x_{n-1}, x_n), d(x_n, x_{n+1}) \right\} = d(x_n, x_{n+1}),
\]

then by (5)

\[
\psi(d(x_n, x_{n+1})) \leq \psi(d(x_n, x_{n+1})) - \varphi(M(x_{n-1}, x_n))
\]

\[
< \psi(d(x_n, x_{n+1})),
\]

which is a contradiction. Hence

\[
\max \left\{ d(x_{n-1}, x_n), d(x_n, x_{n+1}) \right\} = d(x_{n-1}, x_n).
\]
Therefore
\[\psi(d(x_n, x_{n+1})) \leq \psi(d(x_n, x_{n-1})) - \varphi(M(x_{n-1}, x_n)) < \psi(d(x_n, x_{n-1})).\] (6)

Since \(\psi\) is a non-decreasing mapping, then \(\{d(x_n, x_{n+1}) : n \in \mathbb{N} \cup \{0\}\}\) is a non-increasing sequence of positive numbers. Then there exists \(r \geq 0\) such that
\[\lim_{n \to \infty} d(x_n, x_{n+1}) = r.\]

Letting \(n \to \infty\) in (6), we have
\[\psi(r) \leq \psi(r) - \varphi(\lim_{n \to \infty} M(x_{n-1}, x_n)) \leq \psi(r).\]

Therefore \(\varphi(\lim_{n \to \infty} M(x_{n-1}, x_n)) = 0\) and hence \(r = 0\), i.e.,
\[\lim_{n \to \infty} d(x_n, x_{n+1}) = 0.\] (7)

Now, we show that \(\{x_n\}\) is a \(b\)-Cauchy sequence in \(X\). Assume the contrary, that \(\{x_n\}\) is not a \(b\)-Cauchy sequence. Then there exists \(\varepsilon > 0\) and two subsequences \(\{x_{m_i}\}\) and \(\{x_{n_i}\}\) of \(\{x_n\}\) such that \(n_i\) is the smallest index for which
\[n_i > m_i > i \text{ and } d(x_{m_i}, x_{n_i}) \geq \varepsilon.\] (8)

That is
\[d(x_{m_i}, x_{n_i-1}) < \varepsilon.\] (9)

By using (8), (9) and the triangular inequality
\[
\varepsilon \leq d(x_{m_i}, x_{n_i}) \\
\leq sd(x_{m_i}, x_{m_i-1}) + sd(x_{m_i-1}, x_{n_i}) \\
\leq sd(x_{m_i}, x_{m_i-1}) + s^2d(x_{m_i-1}, x_{n_i-1}) + s^2d(x_{n_i-1}, x_{n_i}).
\]

Now, using (7) and taking the upper limit as \(i \to \infty\)
\[\frac{\varepsilon}{s^2} \leq \limsup_{i \to \infty} d(x_{m_i}, x_{n_i-1}).\]

On the other hand
\[d(x_{m_i-1}, x_{n_i-1}) \leq sd(x_{m_i-1}, x_{m_i}) + sd(x_{m_i}, x_{n_i-1}).\]

Using (7), (9) and taking the upper limit as \(i \to \infty\)
\[\limsup_{i \to \infty} d(x_{m_i-1}, x_{n_i-1}) \leq \varepsilon s.\]
Hence
\[
\frac{\varepsilon}{s^2} \leq \limsup_{i \to \infty} \{d(x_{m-1}, x_{n-1}): x_m, x_n \} \leq \varepsilon s. \tag{10}
\]

Again using the triangular inequality
\[
d(x_{m-1}, x_n) \leq sd(x_{m-1}, x_{n-1}) + sd(x_{n-1}, x_n), \tag{11}
\]
\[
\varepsilon \leq d(x_m, x_n) \leq sd(x_m, x_{n-1}) + sd(x_{n-1}, x_n) \tag{12}
\]
and
\[
\varepsilon \leq d(x_m, x_n) \leq sd(x_m, x_{n-1}) + sd(x_{n-1}, x_n). \tag{13}
\]

Using (7) and (10) and taking the upper limit as \(i \to \infty\) in (11) and (12) we get
\[
\frac{\varepsilon}{s} \leq \limsup_{i \to \infty} \{d(x_{m-1}, x_{n-1}) \} \leq \varepsilon s^2. \tag{14}
\]

Again using (7) and (9) and taking the upper limit as \(i \to \infty\) in (13)
\[
\frac{\varepsilon}{s} \leq \limsup_{i \to \infty} \{d(x_m, x_{n-1}) \} \leq \varepsilon. \tag{15}
\]

Since \(\alpha(x_{m-1}, x_{n-1}) \geq 1, \lambda(x_{m-1}) \leq 1\) and \(\lambda(x_{n-1}) \leq 1\) then from (iii) we have
\[
\psi(sd(x_m, x_n)) = \psi(sd(Tx_{m-1}, Tx_{n-1})) \\
\leq \lambda(x_{m-1}) \lambda(x_{n-1}) \left[ \psi(M(x_{m-1}, x_{n-1})) - \varphi(M(x_{m-1}, x_{n-1})) \right] \\
+ \theta \left( d(x_{m-1}, Tx_{m-1}), d(x_{n-1}, Tx_{n-1}), d(x_{m-1}, Tx_{n-1}), d(x_{n-1}, Tx_{m-1}) \right) \tag{16}
\]
where
\[
M(x_{m-1}, x_{n-1}) = \max \left\{ \frac{d(x_{m-1}, x_{n-1})}{2s}, -d(x_{m-1}, Tx_{n-1}), -d(x_{n-1}, Tx_{m-1}), -d(x_{m-1}, Tx_{n-1}), -d(x_{n-1}, Tx_{m-1}) \right\} \tag{17}
\]
and
\[
\theta \left( d(x_{m-1}, Tx_{m-1}), d(x_{n-1}, Tx_{n-1}), d(x_{m-1}, Tx_{n-1}), d(x_{n-1}, Tx_{m-1}) \right) = \theta \left( d(x_{m-1}, x_m), d(x_{n-1}, x_n), d(x_{m-1}, x_n), d(x_{n-1}, x_m) \right). \tag{18}
\]
Taking the upper limit as $i \to \infty$ in (17) and (18) and using (7), (10), (14) and (15) we get
\[
\frac{\varepsilon}{s^2} = \min \left\{ \frac{\varepsilon}{s^2}, \frac{s^2}{2s} \right\} \leq \limsup_{i \to \infty} M(x_{m_i-1}, x_{n_i-1})
\]
\[
= \max \{ \limsup_{i \to \infty} d(x_{m_i-1}, x_{n_i}), 0, 0 \},
\]
\[
\limsup_{i \to \infty} d(x_{m_i-1}, x_{n_i}) + \limsup_{i \to \infty} d(x_{n_i}, x_{m_i})
\]
\[
\leq \max \left\{ \frac{\varepsilon s^2 + \varepsilon}{2s} \right\} = \varepsilon s.
\]
So
\[
\frac{\varepsilon}{s^2} \leq \limsup_{i \to \infty} M(x_{m_i-1}, x_{n_i-1}) \leq \varepsilon s, \tag{19}
\]
and
\[
\limsup_{i \to \infty} \left( d(x_{m_i-1}, T x_{m_i-1}), d(x_{n_i-1}, T x_{n_i-1}), d(x_{m_i-1}, T x_{n_i-1}), d(x_{n_i-1}, T x_{m_i-1}) \right)
\]
\[
= \limsup_{i \to \infty} \left( d(x_{m_i-1}, x_{m_i}), d(x_{n_i-1}, x_{n_i}), d(x_{m_i-1}, x_{n_i}), d(x_{n_i-1}, x_{m_i}) \right) = 0. \tag{20}
\]
Similarly
\[
\frac{\varepsilon}{s^2} \leq \liminf_{i \to \infty} M(x_{m_i-1}, x_{n_i-1}) \leq \varepsilon s. \tag{21}
\]
Now, taking the upper limit as $i \to \infty$ in (16) and using (8), (19) and (20) we have
\[
\psi(\varepsilon s) \leq \psi(\limsup_{i \to \infty} d(x_{m_i}, x_{n_i}))
\]
\[
\leq \psi(\limsup_{i \to \infty} M(x_{m_i-1}, x_{n_i-1})) - \liminf_{n \to \infty} \varphi(M(x_{m_i-1}, x_{n_i-1}))
\]
\[
\leq \psi(\varepsilon s) - \varphi(\liminf_{i \to \infty} M(x_{m_i-1}, x_{n_i-1})),
\]
which implies
\[
\varphi(\liminf_{i \to \infty} M(x_{m_i-1}, x_{n_i-1})) = 0,
\]
so $\liminf_{i \to \infty} M(x_{m_i-1}, x_{n_i-1}) = 0$, which is a contradiction with (21). So $\{x_{n+1}\}$ is a $b$-Cauchy sequence in $X$. Since $X$ is a complete $b$-metric space, there exists $x^* \in X$ such that $x_n \to x^*$ as $n \to \infty$. Also, from (ii) we know $T$ is an $\alpha$-continuous mapping. Hence $Tx_n \to Tx^*$ as $n \to \infty$. Then
\[
d(x^*, Tx^*) \leq sd(x^*, Tx_n) + sd(Tx_n, Tx^*).\]
Letting \( n \to \infty \) in the above inequality
\[
d(x^*, Tx^*) \leq s \lim_{n \to \infty} d(x^*, Tx_n) + s \lim_{n \to \infty} d(Tx_n, Tx^*) = 0.
\]
So \( Tx^* = x^* \).

For self-mappings that are not continuous or \( \alpha \)-continuous we have the following result.

**Theorem 2.2** Let \( (X, d, s) \) be a complete \( b \)-metric space, \( T \) be a self-mapping on \( X \) and \( \alpha : X \times X \to [0, \infty) \) and \( \lambda : X \to [0, +\infty) \) be two functions. Suppose that the following assertions hold.

(i) There exists \( x_0 \in X \) such that \( \alpha(x_0, Tx_0) \geq 1 \) and \( \lambda(x_0) \leq 1 \).

(ii) \( T \) is a triangular \( \alpha \)-admissible and semi \( \lambda \)-subadmissible mapping.

(iii) For all \( x, y \in X \) with \( \alpha(x, y) \geq 1 \)
\[
\psi(sd(Tx, Ty)) \leq \lambda(x)\lambda(y)\left(\psi(M(x, y)) - \varphi(M(x, y))\right) + \theta\left(d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\right),
\]
where \( \psi, \varphi \in \Psi, \theta \in \Theta \) and
\[
M(x, y) = \max\left\{d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty) + d(y, Tx)}{2s}\right\}.
\]

(v) If \( \{x_n\} \) be a sequence such that \( \alpha(x_n, x_{n+1}) \geq 1 \), \( \lambda(x_n) \leq 1 \) for all \( n \in \mathbb{N} \cup \{0\} \) and \( x_n \to x \) as \( n \to \infty \), then \( \alpha(x_n, x) \geq 1 \) for all \( n \in \mathbb{N} \cup \{0\} \) and \( \lambda(x) \leq 1 \).

Then \( T \) has a fixed point.

**Proof.** Let \( x_0 \in X \) be such that \( \alpha(x_0, Tx_0) \geq 1 \) and \( \lambda(x_0) \leq 1 \). Define a sequence \( \{x_n\} \) in \( X \) by \( x_n = T^n x_0 = Tx_{n-1} \) for all \( n \in \mathbb{N} \). Following the proof of the Theorem 2.1, we obtain that \( \{x_n\} \) is a \( b \)-Cauchy sequence such that \( \alpha(x_n, x_{n+1}) \geq 1 \) and \( \lambda(x_n) \leq 1 \) for all \( n \in \mathbb{N} \cup \{0\} \). Since \( X \) is complete, there exists \( x^* \in X \) such that the sequence \( \{x_n\} \) \( b \)-converges to \( x^* \). Using the assumption (v), we have \( \alpha(x_n, x^*) \geq 1 \) for all \( n \in \mathbb{N} \cup \{0\} \) and \( \lambda(x^*) \leq 1 \). By (iii)
\[
\psi(sd(x_{n+1}, Tx^*)) = \psi(sd(Tx_n, Tx^*))
\leq \lambda(x_n)\lambda(x^*)\left[\psi(M(x_n, x^*)) - \varphi(M(x_n, x^*))\right]
+ \theta\left(d(x_n, Tx_n), d(x^*, Tx^*), d(x_n, Tx^*), d(x^*, Tx_n)\right)
\leq \psi(M(x_n, x^*)) - \varphi(M(x_n, x^*))
+ \theta\left(d(x_n, Tx_n), d(x^*, Tx^*), d(x_n, Tx^*), d(x^*, Tx_n)\right),
\]
where
\[
M(x_n, x^*) = \max\left\{d(x_n, x^*), d(x_n, Tx_n), d(x^*, Tx^*), \frac{d(x_n, Tx^*) + d(Tx_n, x^*)}{2s}\right\}
= \max\left\{d(x_n, x^*), d(x_n, x_{n+1}), d(x^*, Tx^*), \frac{d(x_n, Tx^*) + d(x_{n+1}, x^*)}{2s}\right\}.
\]
and

\[
\theta \left( d(x_n, T x_n), d(x^*, T x^*), d(x_n, T x^*), d(x^*, T x_n) \right) \\
= \theta \left( d(x_n, x_{n+1}), d(x^*, T x^*), d(x_n, T x^*), d(x^*, x_{n+1}) \right).
\]

(24)

Letting \( n \to \infty \) in (23) and (24) and using lemma 1.4, we get

\[
\frac{d(x^*, T x^*)}{2s^2} = \min \left\{ d(x^*, T x^*), \frac{d(x^*, T x^*)}{2s^2} \right\} \leq \limsup_{n \to \infty} M(x_n, x^*) \\
\leq \max \left\{ d(x^*, T x^*), \frac{sd(x^*, T x^*)}{2s} \right\} = d(x^*, T x^*),
\]

(25)

and

\[
\theta \left( d(x_n, T x_n), d(x^*, T x^*), d(x_n, T x^*), d(x^*, T x_n) \right) \to 0 \text{ as } n \to \infty.
\]

Similarly

\[
\frac{d(x^*, T x^*)}{2s^2} \leq \liminf_{n \to \infty} M(x_n, x^*) \leq d(x^*, T x^*). \tag{26}
\]

Again, taking the upper limit as \( i \to \infty \) in (22) and using lemma 1.4 and (25) we get

\[
\psi \left( d(x^*, T x^*) \right) = \psi \left( s \frac{d(x^*, T x^*)}{2s} \right) \leq \psi \left( \limsup_{n \to \infty} d(x^*, T x^*) \right) \\
\leq \psi \left( \limsup_{n \to \infty} M(x_n, x^*) \right) - \liminf_{n \to \infty} \varphi \left( M(x_n, x^*) \right) \\
\leq \psi \left( d(x^*, T x^*) \right) - \varphi \left( \liminf_{n \to \infty} M(x_n, x^*) \right).
\]

Hence, \( \varphi \left( \liminf_{n \to \infty} M(x_n, x^*) \right) = 0 \). Then, \( \liminf_{n \to \infty} M(x_n, x^*) = 0 \) which is a contradiction. So, \( x^* = T x^* \).

Example 2.3 Let \( X = \mathbb{R} \) be endowed with the \( b \)-metric

\[
d(x, y) = \begin{cases} 
|x + |y|)^2, & \text{if } x \neq y \\
0, & \text{if } x = y
\end{cases}
\]

for all \( x, y \in X \). Define \( T : X \to X, \alpha : X \times X \to [0, \infty) \) and \( \lambda : X \to [0, \infty) \) by

\[
T x = \begin{cases} 
2x^3 + \sin x, & \text{if } x \in (-\infty, 0) \\
\frac{1}{8} x^2, & \text{if } x \in [0, 1) \\
\frac{1}{8} x, & \text{if } x \in [1, 2) \\
\frac{1}{4}, & \text{if } x \in [2, +\infty)
\end{cases}
\]

and

\[
\alpha(x, y) = \begin{cases} 
2, & \text{if } x, y \in [0, +\infty) \\
0, & \text{otherwise}
\end{cases}
\]

for \( i \to \infty \) in (22) and using lemma 1.4 and (25) we get
and \( \lambda(x) = \begin{cases} 
1, & \text{if } x \in [0, +\infty) \\
2x^2 + 3, & \text{otherwise.} 
\end{cases} \)

Also, define \( \psi, \varphi : [0, \infty) \to [0, +\infty) \) and \( \theta : [0, +\infty)^4 \to [0, +\infty) \) by \( \psi(t) = t, \varphi(t) = \frac{2}{3}t \) and \( \theta(t_1, t_2, t_3, t_4) = \min\{t_1, t_2, t_3, t_4\} \). Clearly \((X, d, s)\) with \( s = 2 \) is a complete \( b\)-metric space, \( \psi, \varphi \in \Psi \) and \( \theta \in \Theta \). Let \( \alpha(x, y) \geq 1 \), then \( x, y \in [0, +\infty) \). On the other hand, \( Tw \in [0, +\infty) \) for all \( w \in [0, +\infty) \). Then \( \alpha(Tx, Ty) \geq 1 \). That is, \( T \) is an \( \alpha \)-admissible mapping. Let \( \alpha(x, y) \geq 1 \) and \( \alpha(y, z) \geq 1 \). So \( x, y, z \in [0, +\infty) \) i.e., \( \alpha(x, z) \geq 1 \). Hence \( T \) is a triangular \( \alpha \)-admissible mapping. Also, let \( \lambda(x) \leq 1 \). Thus \( x \in [0, +\infty) \). That is, \( \lambda(Tx) \leq 1 \). Thus \( T \) is a semi \( \lambda \)-subadmissible mapping. Let \( \{x_n\} \) be a sequence in \( X \) such that \( \alpha(x_n, x_{n+1}) \geq 1 \) and \( \lambda(x_n) \leq 1 \) with \( x_n \to x \) as \( n \to \infty \). Then, \( x_n \in [0, +\infty) \) for all \( n \in \mathbb{N} \). Also \([0, +\infty)\) is a closed set. Then \( x \in [0, +\infty) \). That is \( \alpha(x_n, x) \geq 1 \) for all \( n \in \mathbb{N} \cup \{0\} \) and \( \lambda(x) \leq 1 \). Clearly \( \alpha(0, T0) \geq 1 \) and \( \lambda(0) \leq 1 \).

Let \( \alpha(x, y) \geq 1 \). So \( x, y \in [0, +\infty) \).

Now we consider the following cases:

- Let \( x, y \in [0, 1) \) then

\[
\psi(2d(Tx, Ty)) = 2d(Tx, Ty) = 2\left(\frac{1}{8}x^2 + \frac{1}{8}y^2\right)^2 \\
= \frac{1}{32}(x^2 + y^2)^2 \\
\leq \frac{1}{3}(x + y)^2 \\
= \frac{1}{3}d(x, y) \\
\leq \frac{1}{3}M(x, y) \\
= \psi(M(x, y)) - \varphi(M(x, y)) \\
= \lambda(x)\lambda(y) \left[ \psi(M(x, y)) - \varphi(M(x, y)) \right] + \theta(d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)).
\]

- Let \( x, y \in [1, 2) \) then

\[
\psi(2d(Tx, Ty)) = 2d(Tx, Ty) = 2\left(\frac{1}{8}x + \frac{1}{8}y\right)^2 \\
= \frac{1}{32}(x + y)^2 \\
\leq \frac{1}{3}(x + y)^2 \\
= \frac{1}{3}d(x, y) \\
\leq \frac{1}{3}M(x, y) \\
= \psi(M(x, y)) - \varphi(M(x, y)) \\
\leq \lambda(x)\lambda(y) \left[ \psi(M(x, y)) - \varphi(M(x, y)) \right] \\
+ \theta(d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)).
\]

- Let \( x, y \in [2, \infty) \) then

\[
\psi(2d(Tx, Ty)) = 2d(Tx, Ty) = 2\left(\frac{1}{4}x + \frac{1}{4}y\right)^2 \\
= \frac{1}{4} \leq 1 \\
= \frac{1}{8}(1 + 1)^2 \\
\leq \frac{1}{3}(x + y)^2 \\
= \frac{1}{3}d(x, y) \\
\leq \frac{1}{3}M(x, y) \\
= \psi(M(x, y)) - \varphi(M(x, y)) \\
\leq \lambda(x)\lambda(y) \left[ \psi(M(x, y)) - \varphi(M(x, y)) \right] \\
+ \theta(d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)).
\]
Let $x \in [0, 1)$ and $y \in [1, 2)$ then
\[
\psi(2d(Tx, Ty)) = 2d(Tx, Ty) = 2\left(\frac{1}{8}x^2 + \frac{1}{2}y\right)^2 \\
\leq 2\left(\frac{1}{8}x + \frac{1}{2}y\right)^2 \\
= \frac{1}{2}x^2 + \frac{1}{4}y^2 \\
\leq \frac{1}{2}(x + y)^2 \\
= \frac{1}{2}d(x, y) \\
\leq \frac{1}{2}M(x, y) \\
= \psi(M(x, y)) - \varphi(M(x, y)) \\
= \lambda(x)\lambda(y)\left[\psi(M(x, y)) - \varphi(M(x, y))\right] + \theta(d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)).
\]

Let $x \in [0, 1)$ and $y \in [1, 2)$ then
\[
\psi(2d(Tx, Ty)) = 2d(Tx, Ty) = 2\left(\frac{1}{8}x^2 + \frac{1}{4}\right)^2 \\
\leq 2\left(\frac{1}{8}x + \frac{1}{2}y\right)^2 \\
= \frac{1}{2}x^2 + \frac{1}{4}y^2 \\
\leq \frac{1}{2}(x + y)^2 \\
= \frac{1}{2}d(x, y) \\
\leq \frac{1}{2}M(x, y) \\
= \psi(M(x, y)) - \varphi(M(x, y)) \\
= \lambda(x)\lambda(y)\left[\psi(M(x, y)) - \varphi(M(x, y))\right] + \theta(d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)).
\]

Let $x \in [1, 2)$ and $y \in [2, \infty)$ then
\[
\psi(2d(Tx, Ty)) = 2d(Tx, Ty) = 2\left(\frac{1}{8}x + \frac{1}{2}\right)^2 \\
\leq 2\left(\frac{1}{8}x + \frac{1}{2}y\right)^2 \\
= \frac{1}{2}x^2 + \frac{1}{4}y^2 \\
\leq \frac{1}{2}(x + y)^2 \\
= \frac{1}{2}d(x, y) \\
\leq \frac{1}{2}M(x, y) \\
= \psi(M(x, y)) - \varphi(M(x, y)) \\
\leq \lambda(x)\lambda(y)\left[\psi(M(x, y)) - \varphi(M(x, y))\right] + \theta(d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)).
\]

Therefore $\alpha(x, y) \geq 1$ implies
\[
\psi(2d(Tx, Ty)) \leq \lambda(x)\lambda(y)\left[\psi(M(x, y)) - \varphi(M(x, y))\right] + \theta(d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx))
\]

Hence, all conditions of Theorem 2.2 holds and $T$ has a fixed point. Here, $x = 0$ is a fixed point of $T$.

**Corollary 2.4** Let $(X, d, s)$ be a complete $b$-metric space, $T$ be a self-mapping on $X$ and \( \alpha : X \times X \to [0, \infty) \) and \( \lambda : X \to [0, +\infty) \) be two functions. Suppose that the following assertions hold.
(i) There exists \( x_0 \in X \) such that \( \alpha(x_0, Tx_0) \geq 1 \) and \( \lambda(x_0) \leq 1 \).

(ii) \( T \) is a triangular \( \alpha \)-admissible and semi \( \lambda \)-subadmissible mapping.

(iii) For all \( x, y \in X \)

\[
\psi(s\alpha(x, y)d(Tx, Ty)) \leq \lambda(x)\lambda(y)\left[\psi(M(x, y)) - \varphi(M(x, y))\right] + \theta\left(d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\right),
\]

where \( \psi, \varphi \in \Psi, \theta \in \Theta \) and

\[
M(x, y) = \max\left\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty) + d(y, Tx)\right\},
\]

(v) If \( \{x_n\} \) is a sequence such that \( \alpha(x_n, x_{n+1}) \geq 1, \lambda(x_n) \leq 1 \) for all \( n \in \mathbb{N} \cup \{0\} \) and \( x_n \to x \) as \( n \to \infty \) then \( \alpha(x_n, x) \geq 1 \) for all \( n \in \mathbb{N} \cup \{0\} \) and \( \lambda(x) \leq 1 \).

Then \( T \) has a fixed point.

**Proof.** Let \( \alpha(x, y) \geq 1 \). Since \( \psi \) is increasing then from (iii)

\[
\psi(s\alpha(x, y)d(Tx, Ty)) \leq \psi(s\alpha(x, y)d(Tx, Ty)) \\
\leq \lambda(x)\lambda(y)\left[\psi(M(x, y)) - \varphi(M(x, y))\right] \\
+ \theta\left(d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\right).
\]

Therefore all conditions of Theorem 2.2 holds and \( T \) has a fixed point. \( \blacksquare \)

If in Corollary 2.4 we take \( \alpha(x, y) = 1 \) for all \( x, y \in X \), then we have the following corollary.

**Corollary 2.5** Let \((X, d, s)\) be a complete \( b \)-metric space and \( T \) be a self-mapping on \( X \) and \( \lambda : X \to [0, +\infty) \) be a function. Suppose that the following assertions hold.

(i) there exists \( x_0 \in X \) such that \( \lambda(x_0) \leq 1 \),

(ii) \( T \) is a semi \( \lambda \)-subadmissible mapping,

(iii) for all \( x, y \in X \) we have

\[
\psi(s(d(Tx, Ty))) \leq \psi(M(x, y)) - \varphi(M(x, y)) + \theta\left(d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\right)
\]

where, \( \psi, \varphi \in \Psi, \theta \in \Theta \) and

\[
M(x, y) = \max\left\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty) + d(y, Tx)\right\},
\]

(v) if \( \{x_n\} \) be a sequence such that \( \lambda(x_n) \leq 1 \) for all \( n \in \mathbb{N} \cup \{0\} \) and \( x_n \to x \) as \( n \to \infty \) then \( \lambda(x) \leq 1 \).

Then \( T \) has a fixed point.
3. Some results in $b$–metric spaces endowed with a graph

In this section, we show that many fixed point results in $b$–metric spaces endowed with a graph $G$ (see [4]) can be deduced easily from our presented theorems.

As in [14], let $(E, d, s)$ be a $b$–metric space and $\Delta$ denotes the diagonal of the Cartesian product of $X \times X$. Consider a directed graph $G$ such that the set $V(G)$ of its vertices coincides with $X$ and the set $E(G)$ of its edges contains all loops, that is $E(G) \supseteq \Delta$. We assume that $G$ has no parallel edges, so we can identify $G$ with the pair $(V(G), E(G))$.

Moreover, we may treat $G$ as a weighted graph, see [15, P.309], by assigning to each edge the distance between its vertices. If $x$ and $y$ are vertices in a graph $G$ then a path in $G$ from $x$ to $y$ of length $N$ ($N \in \mathbb{N}$) is a sequence $\{x_i\}_{i=0}^{N}$ of $N + 1$ vertices such that $x_0 = x$, $x_N = y$ and $(x_i, x_{i+1}) \in E(G)$ for $i = 1, \ldots, N$.

**Definition 3.1** [14] Let $(X, d)$ be a metric space endowed with a graph $G$. We say that a self-mapping $T : X \to X$ is a Banach $G$-contraction or simply a $G$-contraction if $T$ preserves the edges of $G$ that is,

$$
\text{for all } x, y \in X, \quad (x, y) \in E(G) \implies (Tx, Ty) \in E(G)
$$

and $T$ decreases the weights of the edges of $G$ in the following way:

$$
\exists \alpha \in (0, 1) \text{ such that for all } x, y \in X, \quad (x, y) \in E(G) \implies d(Tx, Ty) \leq \alpha d(x, y).
$$

**Definition 3.2** [14] A mapping $T : X \to X$ is called $G$-continuous if given $x \in X$ and sequence $\{x_n\}$

$$
x_n \to x \text{ as } n \to \infty \text{ and } (x_n, x_{n+1}) \in E(G) \text{ for all } n \in \mathbb{N} \text{ imply } Tx_n \to Tx.
$$

**Theorem 3.3** Let $(X, d, s)$ be a complete $b$–metric space endowed with a graph $G$ and $T$ be a self-mapping on $X$. Suppose that the following assertions hold.

(i) there exists $x_0 \in X$ such that $(x_0, Tx_0) \in E(G)$ and $\lambda(x_0) \leq 1$,

(ii) $T$ is $G$-continuous and semi $\lambda$-subadmissible mapping,

(iii) $\forall x, y \in X[(x, y) \in E(G) \implies (T(x), T(y)) \in E(G)]$

(iv) $\forall x, y, z \in X[(x, y) \in E(G) \text{ and } (y, z) \in E(G) \implies (x, z) \in E(G)]$

(v) for all $x, y \in X$ with $(x, y) \in E(G)$ we have,

$$
\psi(sd(Tx, Ty)) \leq \lambda(x)\lambda(y)\left[\psi(M(x, y)) - \varphi(M(x, y))\right] + \theta\left(d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\right)
$$

where, $\psi, \varphi \in \Psi$, $\theta \in \Theta$ and

$$
M(x, y) = \max\left\{d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty) + d(y, Tx)}{2s}\right\}.
$$

Then $T$ has a fixed point.

**Proof.** Define $\alpha : X^2 \to [0, +\infty)$ by

$$
\alpha(x, y) = \begin{cases} 
2, & \text{if } (x, y) \in E(G) \\
\frac{1}{2}, & \text{otherwise}.
\end{cases}
$$
First we show that $T$ is a triangular $\alpha$-admissible mapping. Let $\alpha(x, y) \geq 1$ then $(x, y) \in E(G)$. From (iii) $(Tx, Ty) \in E(G)$. That is $\alpha(Tx, Ty) \geq 1$. Also let $\alpha(x, y) \geq 1$ and $\alpha(y, z) \geq 1$. So $(x, y) \in E(G)$ and $(y, z) \in E(G)$. From (iv) we get $(x, z) \in E(G)$, i.e. $\alpha(x, z) \geq 1$. Thus $T$ is a triangular $\alpha$-admissible mapping. Let $T$ be $G$-continuous. So

$$x_n \to x \text{ as } n \to \infty \text{ and } (x_n, x_{n+1}) \in E(G) \text{ for all } n \in \mathbb{N} \text{ imply } Tx_n \to Tx.$$ 

That is,

$$x_n \to x \text{ as } n \to \infty \text{ and } \alpha(x_n, x_{n+1}) \geq 1 \text{ for all } n \in \mathbb{N} \text{ imply } Tx_n \to Tx$$

which implies that $T$ is $\alpha$-continuous. From (i) there exists $x_0 \in X$ such that $(x_0, Tx_0) \in E(G)$. That is $\alpha(x_0, Tx_0) \geq 1$. Let $\alpha(x, y) \geq 1$ then $(x, y) \in E(G)$. Now from (v) we have

$$\psi(sd(Tx, Ty)) \leq \lambda(x)\lambda(y)\left[\psi(M(x, y)) - \varphi(M(x, y))\right] + \theta\left(d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\right)$$

Hence all conditions of Theorem 2.1 are satisfied and $T$ has a fixed point. 

In Theorem 3.3 we take $\theta(t_1, t_2, t_3, t_4) = \min\{t_1, t_2, t_3, t_4\}$.

**Corollary 3.4** Let $(X, d, s)$ be a complete $b$-metric space endowed with a graph $G$ and $T$ be a self-mapping on $X$. Suppose that the following assertions hold.

(i) there exists $x_0 \in X$ such that $(x_0, Tx_0) \in E(G)$ and $\lambda(x_0) \leq 1$,

(ii) $T$ is $G$-continuous and semi $\lambda$-subadmissible mapping,

(iii) $\forall x, y \in X[(x, y) \in E(G) \Rightarrow (T(x), T(y)) \in E(G)]$

(iv) $\forall x, y, z \in X[(x, y) \in E(G) \text{ and } (y, z) \in E(G) \Rightarrow (x, z) \in E(G)]$

(v) for all $x, y \in X$ with $(x, y) \in E(G)$ we have,

$$\psi(sd(Tx, Ty)) \leq \lambda(x)\lambda(y)\left[\psi(M(x, y)) - \varphi(M(x, y))\right] + L \min\{d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\}$$

where, $\psi, \varphi \in \Psi$, $L \geq 0$ and

$$M(x, y) = \max\left\{d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty) + d(y, Tx)}{2s}\right\}.$$ 

Then $T$ has a fixed point.

**Theorem 3.5** Let $(X, d, s)$ be a complete $b$-metric space endowed with a graph $G$ and $T$ be a self-mapping on $X$. Suppose that the following assertions hold.

(i) there exists $x_0 \in X$ such that $(x_0, Tx_0) \in E(G)$ and $\lambda(x_0) \leq 1$,

(ii) $T$ is semi $\lambda$-subadmissible mapping,

(iii) $\forall x, y \in X[(x, y) \in E(G) \Rightarrow (T(x), T(y)) \in E(G)]$

(iv) $\forall x, y, z \in X[(x, y) \in E(G) \text{ and } (y, z) \in E(G) \Rightarrow (x, z) \in E(G)]$

(v) for all $x, y \in X$ with $(x, y) \in E(G)$ we have,

$$\psi(sd(Tx, Ty)) \leq \lambda(x)\lambda(y)\left[\psi(M(x, y)) - \varphi(M(x, y))\right] + \theta\left(d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\right)$$

(29)
where, \((\psi, \varphi \in \Psi), \theta \in \Theta\) and

\[
M(x,y) = \max \left\{ d(x,y), d(x,Tx), d(y,Ty), \frac{d(x,Ty)+d(y,Tx)}{2s} \right\}.
\]

(vi) if \(\{x_n\}\) be a sequence in \(X\) such that \((x_n, x_{n+1}) \in E(G), \lambda(x_n) \leq 1\) for all \(n \in \mathbb{N} \cup \{0\}\) and \(x_n \to x\) as \(n \to \infty\) then \((x_n, x) \in E(G)\) for all \(n \in \mathbb{N} \cup \{0\}\) and \(\lambda(x) \leq 1\).

Then \(T\) has a fixed point.

**Proof.** Define the mapping \(\alpha : X^2 \to [0, +\infty)\) as in the proof of Theorem 3.3. Similar to the proof of Theorem 3.3 we can prove that the conditions (i)-(iii) of Theorem 2.2 are satisfied. Let \(\{x_n\}\) be a sequence in \(X\) such that \(\alpha(x_n, x_{n+1}) \geq 1\) and \(\lambda(x_n) \leq 1\) for all \(n \in \mathbb{N} \cup \{0\}\) and \(x_n \to x\) as \(n \to \infty\). Then \((x_n, x_{n+1}) \in E(G)\) and \(\lambda(x_n) \leq 1\) for all \(n \in \mathbb{N} \cup \{0\}\). From (vi) we get \((x_n, x) \in E(G)\) and \(\lambda(x) \leq 1\). That is \(\alpha(x_n, x) \geq 1\) for all \(n \in \mathbb{N} \cup \{0\}\) and \(\lambda(x) \leq 1\). Therefore all conditions of Theorem 2.2 holds and \(T\) has a fixed point.

**Corollary 3.6** Let \((X, d, s)\) be a complete \(b\)-metric space endowed with a graph \(G\) and \(T\) be a self-mapping on \(X\). Suppose that the following assertions hold.

(i) there exists \(x_0 \in X\) such that \((x_0, Tx_0) \in E(G)\) and \(\lambda(x_0) \leq 1\),
(ii) \(T\) is semi-\(\lambda\)-subadmissible mapping,
(iii) \(\forall x, y \in X, (x,y) \in E(G) \Rightarrow (T(x), T(y)) \in E(G)\)
(iv) \(\forall x, y, z \in X, (x,y) \in E(G)\) and \((y,z) \in E(G) \Rightarrow (x,z) \in E(G)\)
(v) for all \(x, y \in X\) with \((x, y) \in E(G)\) we have,

\[
\psi(sd(Tx,Ty)) \leq \lambda(x)\lambda(y)\left[\psi(M(x,y)) - \varphi(M(x,y))\right] + L \min\{d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx)\}
\]

where, \((\psi, \varphi \in \Psi), L \geq 0\) and

\[
M(x,y) = \max \left\{ d(x,y), d(x,Tx), d(y,Ty), \frac{d(x,Ty)+d(y,Tx)}{2s} \right\}.
\]

(vi) if \(\{x_n\}\) be a sequence in \(X\) such that \((x_n, x_{n+1}) \in E(G), \lambda(x_n) \leq 1\) for all \(n \in \mathbb{N} \cup \{0\}\) and \(x_n \to x\) as \(n \to \infty\), then \((x_n, x) \in E(G)\) for all \(n \in \mathbb{N} \cup \{0\}\) and \(\lambda(x) \leq 1\).

Then \(T\) has a fixed point.

### 4. Some results in \(b\)-metric spaces endowed with a partial ordered

The existence of fixed points in partially ordered sets has been considered by many authors (such as [19], [21–26] and [29] etc.). Later on, some generalizations of [26] are given in [27]. Several applications of these results to matrix equations are presented in [26].

Let \(X\) be a nonempty set. If \((X, d, s)\) is a \(b\)-metric space and \((X, \leq)\) be a partially ordered set, then \((X, d, s, \leq)\) is called an ordered \(b\)-metric space. Two elements \(x, y \in X\) are called comparable if \(x \leq y\) or \(y \leq x\) hold. A mapping \(T : X \to X\) is said to be non-decreasing if \(x \leq y\) implies \(Tx \leq Ty\) for all \(x, y \in X\).

In this section, we will show that many fixed point results in partially ordered \(b\)-metric spaces can be deduced easily from our obtained results.
Theorem 4.1 Let \((X, d, s, \preceq)\) be a complete ordered \(b\)-metric space and \(T\) be a self-mapping on \(X\). Suppose that the following assertions hold.

(i) there exists \(x_0 \in X\) such that \(x_0 \preceq Tx_0\) and \(\lambda(x_0) \leq 1\),

(ii) \(T\) is continuous and semi \(\lambda\)-subadmissible mapping,

(iii) \(T\) is an increasing mapping,

(v) for all \(x, y \in X\) with \(x \preceq y\) we have,

\[
\psi(sd(Tx, Ty)) \leq \lambda(x)\lambda(y) \left[\psi(M(x, y)) - \varphi(M(x, y))\right] + \theta\left(d(x,Tx),d(y,Ty),d(x,Ty),d(y,Tx)\right)
\]

where, \(\psi, \varphi \in \Psi, \theta \in \Theta\) and

\[
M(x, y) = \max \left\{ d(x, y), d(x, Tx), d(y, Ty), \frac{d(x,Ty) + d(y,Tx)}{2s} \right\}.
\]

Then \(T\) has a fixed point.

Proof. Define \(\alpha : X^2 \to [0, +\infty)\) by

\[
\alpha(x, y) = \begin{cases} 
2, & \text{if } x \preceq y \\
\frac{1}{2}, & \text{otherwise}
\end{cases}
\]

First, we prove that \(T\) is a triangular \(\alpha\)-admissible mapping. Let \(\alpha(x, y) \geq 1\), then \(x \preceq y\).

Since \(T\) is increasing, then we have \(Tx \preceq Ty\). That is, \(\alpha(Tx, Ty) \geq 1\). Suppose that \(\alpha(x, y) \geq 1\) and \(\alpha(y, z) \geq 1\). Then \(x \preceq y\) and \(y \preceq z\). Hence \(x \preceq z\) i.e., \(\alpha(x, z) \geq 1\).

Therefore, \(T\) is a triangular \(\alpha\)-admissible mapping. Since \(T\) is continuous then it is \(\alpha\)-continuous too. From (i) there exists \(x_0 \in X\) such that \(\alpha(x_0, Tx_0) \geq 1\). That is, \(\alpha(x_0, Tx_0) \geq 1\). Let \(\alpha(x, y) \geq 1\), then \(x \preceq y\). Now, from (v) we have

\[
\psi(sd(Tx, Ty)) \leq \lambda(x)\lambda(y) \left[\psi(M(x, y)) - \varphi(M(x, y))\right] + \theta\left(d(x,Tx),d(y,Ty),d(x,Ty),d(y,Tx)\right).
\]

Hence, all conditions of Theorem 2.1 are satisfied and \(T\) has a fixed point. \(\square\)

If in Theorem 3.3 we take \(\theta(t_1, t_2, t_3, t_4) = L\psi(\min\{t_1, t_4\})\) where \(L \geq 0\), then we have the following Corollary.

Corollary 4.2 Let \((X, d, s, \preceq)\) be a complete ordered \(b\)-metric space and \(T\) be a self-mapping on \(X\). Suppose that the following assertions hold.

(i) there exists \(x_0 \in X\) such that \(x_0 \preceq Tx_0\) and \(\lambda(x_0) \leq 1\),

(ii) \(T\) is continuous and semi \(\lambda\)-subadmissible mapping,

(iii) \(T\) is an increasing mapping,

(v) for all \(x, y \in X\) with \(x \preceq y\) we have,

\[
\psi(sd(Tx, Ty)) \leq \lambda(x)\lambda(y) \left[\psi(M(x, y)) - \varphi(M(x, y))\right] + L\psi(\min\{d(x,Tx),d(y,Tx)\})
\]

where, \(\psi, \varphi \in \Psi, L \geq 0\) and

\[
M(x, y) = \max \left\{ d(x, y), d(x, Tx), d(y, Ty), \frac{d(x,Ty) + d(y,Tx)}{2s} \right\}.
\]
Then $T$ has a fixed point.

If in Corollary 3.3 we take $\lambda(x) = 1$ for all $x \in X$, then we have the following Corollary.

**Corollary 4.3** [27, Theorem 3] Let $(X, d, s, \leq)$ be a complete ordered $b$-metric space and $T$ be a self-mapping on $X$. Suppose that the following assertions hold.

(i) there exists $x_0 \in X$ such that $x_0 \leq Tx_0$,

(ii) $T$ is continuous,

(iii) $T$ is an increasing mapping,

(iv) for all $x, y \in X$ with $x \leq y$ we have,

$$\psi(sd(Tx, Ty)) \leq \psi(M(x, y)) - \varphi(M(x, y)) + L\psi(\min\{d(x, Tx), d(y, Ty)\})$$

where, $\psi, \varphi \in \Psi$, $L \geq 0$ and

$$M(x, y) = \max\left\{d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty) + d(y, Tx)}{2s}\right\}.$$  

Then $T$ has a fixed point.

**Theorem 4.4** Let $(X, d, s, \leq)$ be a complete partially ordered $b$-metric space and let $T$ be a self-mapping on $X$. Suppose that the following assertions hold.

(i) there exists $x_0 \in X$ such that $x_0 \leq Tx_0$ and $\lambda(x_0) \leq 1$,

(ii) $T$ is a semi $\lambda$-subadmissible mapping,

(iii) $T$ is an increasing mapping,

(iv) for all $x, y \in X$ with $x \leq y$ we have,

$$\psi(sd(Tx, Ty)) \leq \lambda(x)\lambda(y)\left[\psi(M(x, y)) - \varphi(M(x, y))\right] + \theta\left(d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\right)$$

where, $(\psi, \varphi \in \Psi)$, $\theta \in \Theta$ and

$$M(x, y) = \max\left\{d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty) + d(y, Tx)}{2s}\right\}.$$  

(v) if $\{x_n\}$ be an increasing sequence in $X$ such that $\lambda(x_n) \leq 1$ for all $n \in \mathbb{N} \cup \{0\}$ and $x_n \to x$ as $n \to \infty$ then $x_n \leq x$ for all $n \in \mathbb{N} \cup \{0\}$ and $\lambda(x) \leq 1$.

Then $T$ has a fixed point.

**Proof.** Define the mapping $\alpha : X^2 \to [0, +\infty)$ as in the proof of Theorem 3.3. Analogous to the proof of Theorem 3.3 we can prove all the conditions (i)-(iii) of Theorem 2.2 are satisfied. Let $\{x_n\}$ be a sequence in $X$ such that $\alpha(x_n, x_{n+1}) \geq 1$ and $\lambda(x_n) \leq 1$ for all $n \in \mathbb{N} \cup \{0\}$ and $x_n \to x$ as $n \to \infty$. Then $x_n \leq x_{n+1}$ and $\lambda(x_n) \leq 1$ for all $n \in \mathbb{N} \cup \{0\}$. From (v) we get, $x_n \leq x$ and $\lambda(x) \leq 1$. That is, $\alpha(x_n, x) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$ and $\lambda(x) \leq 1$. Therefore all conditions of Theorem 2.2 holds and $T$ has a fixed point. 

**Corollary 4.5** Let $(X, d, s, \leq)$ be a complete partially ordered $b$-metric space and $T$ be a self-mapping on $X$. Suppose that the following assertions hold.

(i) there exists $x_0 \in X$ such that, $x_0 \leq Tx_0$ and $\lambda(x_0) \leq 1$,

(ii) $T$ is a semi $\lambda$-subadmissible mapping,

(iii) $T$ is an increasing mapping,
(iv) for all \( x, y \in X \) with \( x \preceq y \) we have,
\[
\psi(sd(Tx, Ty)) \leq \lambda(x)\lambda(y)\left[\psi(M(x, y)) - \varphi(M(x, y))\right] + L\psi(\min\{d(x, Tx), d(y, Tx)\})
\]
where, \( \psi, \varphi \in \Psi, \theta \in \Theta \) and
\[
M(x, y) = \max\left\{ d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty) + d(y, Tx)}{2s} \right\}.
\]

(v) if \( \{x_n\} \) be an increasing sequence in \( X \) such that \( \lambda(x_n) \leq 1 \) for all \( n \in \mathbb{N} \setminus \{0\} \) and \( x_n \to x \) as \( n \to \infty \), then \( x_n \preceq x \) for all \( n \in \mathbb{N} \cup \{0\} \) and \( \lambda(x) \leq 1 \).

Then \( T \) has a fixed point.

**Corollary 4.6** [27, Theorem 4] Let \( (X, d, s, \preceq) \) be a complete partially ordered \( b \)-metric space and \( T \) be a self-mapping on \( X \). Suppose that the following assertions hold.

(i) there exists \( x_0 \in X \) such that \( x_0 \preceq Tx_0 \),

(ii) \( T \) is an increasing mapping,

(iv) for all \( x, y \in X \) with \( x \preceq y \) we have,
\[
\psi(sd(Tx, Ty)) \leq \psi(M(x, y)) - \varphi(M(x, y)) + L\psi(\min\{d(x, Tx), d(y, Tx)\})
\]
where, \( (\psi, \varphi) \in \Psi \), \( L \geq 0 \) and
\[
M(x, y) = \max\left\{ d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty) + d(y, Tx)}{2s} \right\}.
\]

(v) if \( \{x_n\} \) be an increasing sequence in \( X \) such that \( x_n \to x \) as \( n \to \infty \) then \( x_n \preceq x \) for all \( n \in \mathbb{N} \cup \{0\} \).

Then \( T \) has a fixed point.

### 5. Some integral type contractions

Let \( \Phi \) denotes the set of all functions \( \phi : [0, +\infty) \to [0, +\infty) \) satisfying the following properties:

- every \( \phi \in \Phi \) is a Lebesgue integrable function on each compact subset of \([0, +\infty)\),

- for any \( \phi \in \Phi \) and any \( \epsilon > 0 \), \( \int_0^\epsilon \phi(\tau)d\tau > 0 \).

Note that if we take \( \psi(t) = \int_0^t \phi(\tau)d\tau \) where \( \phi \in \Phi \) then \( \psi \in \Psi \).

Also note that if \( \psi \in \Psi \) and \( \theta \in \Theta \) then \( \psi\theta \in \Theta \).

If in Theorem 2.1 we take \( \psi(t) = \int_0^t \phi(\tau)d\tau \), \( \varphi(t) = (1 - r) \int_0^t \phi(\tau)d\tau \) for all \( t \in [0, \infty) \)
where \( 0 \leq r < 1 \) and replace \( \theta \) by \( \psi\theta \) then we have the following theorem.

**Theorem 5.1** Let \( (X, d, s) \) be a complete \( b \)-metric space, \( T \) be a self-mapping on \( X \) and \( \alpha : X \times X \to [0, \infty) \) and \( \lambda : X \to [0, +\infty) \) be two functions. Suppose that the following assertions hold.

(i) there exists \( x_0 \in X \) such that \( \alpha(x_0, Tx_0) \geq 1 \) and \( \lambda(x_0) \leq 1 \),

(ii) \( T \) is \( \alpha \)-continuous, triangular \( \alpha \)-admissible and semi \( \lambda \)-subadmissible mapping,
(iii) for all \(x, y \in X\) with \(\alpha(x, y) \geq 1\) we have

\[
\int_0^{d(Tx,Ty)} \phi(\tau) d\tau \leq \frac{r \lambda(x) \lambda(y)}{s} \int_0^{M(x,y)} \phi(\tau) d\tau + \int_0^{\Theta(d(x,Tx),d(y,Ty),d(x,Ty),d(y,Tx))} \phi(\tau) d\tau
\]

where, \(0 \leq r < 1, \phi \in \Phi, \theta \in \Theta\) and

\[
M(x,y) = \max \left\{ d(x,y), d(x,Tx), d(y,Ty), \frac{d(x,Ty) + d(y,Tx)}{2s} \right\}.
\]

Then \(T\) has a fixed point.

**Theorem 5.2** Let \((X,d)\) be a complete \(b\)-metric space, \(T\) be a self-mapping on \(X\) and \(\alpha : X \times X \to [0, \infty)\) and \(\lambda : X \to [0, \infty)\) be two functions. Suppose that the following assertions hold.

(i) there exists \(x_0 \in X\) such that, \(\alpha(x_0, Tx_0) \geq 1\) and \(\lambda(x_0) \leq 1\),

(ii) \(T\) is a triangular \(\alpha\)-admissible and semi \(\lambda\)-subadmissible mapping,

(iii) for all \(x, y \in X\) with \(\alpha(x, y) \geq 1\) we have

\[
\int_0^{d(Tx,Ty)} \phi(\tau) d\tau \leq \frac{r \lambda(x) \lambda(y)}{s} \int_0^{M(x,y)} \phi(\tau) d\tau + \int_0^{\Theta(d(x,Tx),d(y,Ty),d(x,Ty),d(y,Tx))} \phi(\tau) d\tau
\]

where, \(0 \leq r < 1, \phi \in \Phi, \theta \in \Theta\) and

\[
M(x,y) = \max \left\{ d(x,y), d(x,Tx), d(y,Ty), \frac{d(x,Ty) + d(y,Tx)}{2s} \right\}.
\]

Then \(T\) has a fixed point.

**Theorem 5.3** Let \((X,d)\) be a complete \(b\)-metric space endowed with a graph \(G\) and \(T\) be a self-mapping on \(X\). Suppose that the following assertions hold.

(i) there exists \(x_0 \in X\) such that, \((x_0, Tx_0) \in E(G)\) and \(\lambda(x_0) \leq 1\),

(ii) \(T\) is \(G\)-continuous and semi \(\lambda\)-subadmissible mapping,

(iii) \(\forall x, y \in X[(x,y) \in E(G) \Rightarrow (T(x), T(y)) \in E(G)]\)

(iv) \(\forall x, y, z \in X[(x,y) \in E(G) \text{ and } (y,z) \in E(G) \Rightarrow (x,z) \in E(G)]\)

(v) for all \(x, y \in X\) with \((x,y) \in E(G)\) we have,

\[
\int_0^{d(Tx,Ty)} \phi(\tau) d\tau \leq \frac{r \lambda(x) \lambda(y)}{s} \int_0^{M(x,y)} \phi(\tau) d\tau + \int_0^{\Theta(d(x,Tx),d(y,Ty),d(x,Ty),d(y,Tx))} \phi(\tau) d\tau
\]

where, \(0 \leq r < 1, \phi \in \Phi, \theta \in \Theta\) and

\[
M(x,y) = \max \left\{ d(x,y), d(x,Tx), d(y,Ty), \frac{d(x,Ty) + d(y,Tx)}{2s} \right\}.
\]

Then \(T\) has a fixed point.
Theorem 5.4 Let \((X, d, s)\) be a complete \(b\)-metric space endowed with a graph \(G\) and \(T\) be a self-mapping on \(X\). Suppose that the following assertions hold.

(i) there exists \(x_0 \in X\) such that \((x_0, Tx_0) \in E(G)\) and \(\lambda(x_0) \leq 1\),
(ii) \(T\) is semi \(\lambda\)-subadmissible mapping,
(iii) \(\forall x, y \in X \[(x, y) \in E(G) \Rightarrow (T^nx, T^ny) \in E(G)\]\)
(iv) \(\forall x, y, z \in X\[(x, y) \in E(G)\) and \((y, z) \in E(G) \Rightarrow (x, z) \in E(G)\]\)
(v) for all \(x, y \in X\) with \((x, y) \in E(G)\) we have,

\[
\int_0^{d(Tx, Ty)} \phi(\tau) d\tau \leq \frac{r \lambda(x) \lambda(y)}{s} \int_0^{M(x, y)} \phi(\tau) d\tau + \int_0^{g} \left( d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx) \right) \phi(\tau) d\tau
\]

where, \(0 \leq r < 1\), \(\phi \in \Phi\), \(\theta \in \Theta\) and

\[
M(x, y) = \max \left\{ d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty) + d(y, Tx)}{2s} \right\}.
\]

(vi) if \(\{x_n\}\) be a sequence in \(X\) such that \((x_n, x_{n+1}) \in E(G)\), \(\lambda(x_n) \leq 1\) for all \(n \in \mathbb{N} \cup \{0\}\) and \(x_n \to x\) as \(n \to \infty\) then \((x_n, x) \in E(G)\) for all \(n \in \mathbb{N} \cup \{0\}\) and \(\lambda(x) \leq 1\).

Then \(T\) has a fixed point.

Theorem 5.5 Let \((X, d, s, \preceq)\) be a complete ordered \(b\)-metric space and \(T\) be a self-mapping on \(X\). Suppose that the following assertions hold.

(i) there exists \(x_0 \in X\) such that \(x_0 \preceq Tx_0\) and \(\lambda(x_0) \leq 1\),
(ii) \(T\) is continuous and semi \(\lambda\)-subadmissible mapping,
(iii) \(T\) is an increasing mapping,
(v) for all \(x, y \in X\) with \(x \preceq y\) we have

\[
\int_0^{d(Tx, Ty)} \phi(\tau) d\tau \leq \frac{r \lambda(x) \lambda(y)}{s} \int_0^{M(x, y)} \phi(\tau) d\tau + \int_0^{g} \left( d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx) \right) \phi(\tau) d\tau
\]

where, \(0 \leq r < 1\), \(\phi \in \Phi\), \(\theta \in \Theta\) and

\[
M(x, y) = \max \left\{ d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty) + d(y, Tx)}{2s} \right\}.
\]

Then \(T\) has a fixed point.

References


